First we recall some of the basic laws of logarithmic function:

- $\log_b^x = y \quad where \quad b^y = x$,
- $\log_b(mn) = \log_b m + \log_b n$,
- $\log_b \frac{m}{n} = \log_b m - \log_b n$,
- $\log_b m^r = r \log_b m$,
- $\log_b^m = \frac{\log_a^m}{\log_a^b}$,
- $\log_e^x = \ln x$ (here the number e is called the base of the natural logarithm ln x).
- $\log_b^m = \frac{\ln m}{\ln b}$,
- $\log_a^a = 1$.

Rule (1)

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

Example (1) Find $\frac{dy}{dx}$ if $y = 2\ln x$.

Solution

$$y' = 2\frac{d}{dx}\ln x = 2.(\frac{1}{x}) = \frac{2}{x}.$$

Now, consider y = ln u, u = f(x), we need to calculate $\frac{dy}{dx}$,

From the chain rule we have $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$, and $\frac{dy}{du} = \frac{1}{u}$. Thus, $\frac{dy}{dx} = \frac{1}{u}.\frac{du}{dx}$ and we have the following rule:

Rule (2) If y = ln u, u = f(x), then

$$\frac{dy}{dx} = \frac{1}{u}.\frac{du}{dx}.$$

Example (2) If $y = \ln(1 - x^2)$, find $\dfrac{dy}{dx}$.

Solution

Let $u = 1 - x^2$ then y = ln u and $\dfrac{dy}{du} = \dfrac{1}{u}$, $\dfrac{du}{dx} = -2x$,

From rule 2 we have, $\dfrac{dy}{dx} = \dfrac{1}{u}.\dfrac{du}{dx}$ then $\dfrac{dy}{dx} = (\dfrac{1}{u}).(-2x) = -\dfrac{2x}{1 - x^2}$.

Example (3) If $y = \ln(3x - 7)$, find $\dfrac{dy}{dx}$.

Solution

Let $u = 3x - 7$ then y = ln u and $\dfrac{dy}{du} = \dfrac{1}{u}$, $\dfrac{du}{dx} = 3$,

From rule 2 we have, $\dfrac{dy}{dx} = \dfrac{1}{u}.\dfrac{du}{dx}$ then $\dfrac{dy}{dx} = (\dfrac{1}{u}).(3) = \dfrac{3}{3x - 7}$.

Example (4) Find $\dfrac{dy}{dx}$ if $y = \dfrac{x^2 - 1}{\ln x}$.

Solution

$$\frac{dy}{dx} = \frac{d}{dx}(\frac{x^2 - 1}{\ln x}) = \frac{(\ln x)\frac{d}{dx}(x^2 - 1) - (x^2 - 1)\frac{d}{dx}\ln x}{(\ln x)^2}$$

$$= \frac{(\ln x).(2x) - (x^2 - 1)(\frac{1}{x})}{(\ln x)^2} = \frac{2x\ln x - x + \frac{1}{x}}{(\ln x)^2}$$

$$= \frac{2x^2 \ln x - x^2 + 1}{x(\ln x)^2}.$$

Example (5) Find $\frac{dy}{dx}$ if $y = \ln(x^2 + 4x + 5)^3$.

Solution

Let $u = x^2 + 4x + 5$ then $y = \ln u^3$, and this implies that $y = 3\ln u$. Also, $\frac{du}{dx} = 2x + 4$ and

$\frac{dy}{du} = \frac{3}{u}$.

Since $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$ then $\frac{dy}{dx} = \frac{3}{u}(2x+4) = \frac{3(2x+4)}{(x^2+4x+5)}$.

Example (6) Find $\frac{d}{dx}x^2 \ln x$.

Solution

$$\frac{d}{dx}x^2 \ln x = \ln x \frac{d}{dx}x^2 + x^2 \frac{d}{dx}\ln x = (\ln x)(2x) + x^2(\frac{1}{x}) = 2x\ln x + x = x(2\ln x + 1)$$
$$= x(\ln x^2 + 1).$$

Example (7) If $f(l) = \ln(\frac{1+l}{1-l})$, find $\frac{df(l)}{dl}$.

Solution

Let $u = \frac{1+l}{1-l}$ then $f(l) = \ln u$, this implies that $\frac{df}{du} = \frac{1}{u}$, and

$$\frac{du}{dl} = \frac{(1-l)\frac{d}{dl}(1+l) - (1+l)\frac{d}{dl}(1-l)}{(1-l)^2} = \frac{(1-l)(1) - (1+l)(-1)}{(1-l)^2} = \frac{1-l+1+l}{(1-l)^2}$$
$$= \frac{2}{(1-l)^2}.$$

Now, from the chain rule we know that $\frac{df(l)}{dl} = \frac{df(l)}{du}.\frac{du}{dl}$, then

$$\frac{df(l)}{dl} = \frac{1}{u}.\frac{2}{(1-l)^2} = (\frac{1-l}{1+l}).\frac{2}{(1-l)^2} = \frac{2}{(1-l)(1+l)}.$$

Example (8) Differentiate $y = \log_2^x$.

Solution

Notice that, $y = \log_2^x = \dfrac{\ln x}{\ln 2}$, then

$$\frac{dy}{dx} = \frac{d}{dx}(\frac{\ln x}{\ln 2}) = \frac{1}{\ln 2}\frac{d}{dx}(\ln x) = \frac{1}{\ln 2}.\frac{1}{x} = \frac{1}{x\ln 2}.$$

Example (9) Find the marginal-revenue function if the demand function is $p = \dfrac{25}{\ln(q+2)}$

Solution

The marginal revenue function is $\dfrac{dr}{dq}$,

Since r = p . q, then $r = pq = p = \dfrac{25}{\ln(q+2)}(q) = p = \dfrac{25q}{\ln(q+2)}$

$$\therefore \ the\ marginal\ revenue\ \frac{dr}{dq} = \frac{d}{dq}\frac{25q}{\ln(q+2)} = \frac{\ln(q+2).(25) - 25q(\frac{1}{(q+2)})}{[\ln(q+2)]^2}$$

$$= \frac{25\ln(q+2) - \frac{25q}{(q+2)}}{[\ln(q+2)]^2} = \frac{25(q+2)\ln(q+2) - 25q}{(q+2)[\ln(q+2)]^2}.$$

Example (10) A total-cost function is given by:

$$c = 25\ln(q+1) + 12,$$

Find the marginal cost when q = 6.

Solution

The marginal cost is $\dfrac{dc}{dq}$

$$\frac{dc}{dq} = \frac{d}{dq}[25\ln(q+1) + 12] = 25\frac{d}{dq}\ln(q+1) + 0 = \frac{25}{q+1}.$$

When q = 6 the marginal cost is $\left.\dfrac{dc}{dq}\right|_{q=6} = \dfrac{25}{(6)+1} = \dfrac{25}{7}$.

Home work: solve the book page 595, the following problems:

[18] Differentiate $y = x^2 \log_2^x$.

[20] Find the derivative of $y = \dfrac{x^2}{\ln x}$.

[26] If $f(t) = \ln(\dfrac{t^4}{1+t+t^2})$, find $f'(t)$.

[32] If $y = \ln[(5x+2)^4 (8x-3)^6]$, find $\dfrac{dy}{dx}$.

[50] A manufacture's average cost function, in dollars, is given by

$$\bar{c} = \frac{350}{\ln(q+2)},$$

Find the marginal cost (approximated to the nearest two decimal places) when q = 40.