Example (1) Differentiate $y = x^x$

Solution

Here we can't use the power rule, and by taking the natural logarithm of both sides of the equation, then

$$\ln y = \ln x^x$$

$\Rightarrow \quad \ln y = x \ln x$, after that differentiating both sides of this equation with respect to x, we have:

$$\frac{d}{dx}\ln y = \frac{d}{dx} x \ln x$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \ln x . \frac{d}{dx} x + x . \frac{d}{dx}\ln x$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \ln x .(1) + x . \frac{1}{x}$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \ln x . + 1 \Rightarrow \quad \frac{dy}{dx} = y(\ln x . + 1) \ ,$$

And finally, $$\frac{dy}{dx} = x^x(\ln x + 1) \ .$$

Remark we use logarithmic differentiation for functions on the form $u(x)^{v(x)}$, in other words when the base is variables and the power is also variables.

Example (2) Find the derivative of $y = (1 + e^x)^{\ln x}$

Solution

This function has the form $u(x)^{v(x)}$, hence we may use the technique of logarithmic differentiation. By taking the natural logarithm of both sides of the equation, we obtain:

$$\ln y = \ln(1 + e^x)^{\ln x}$$

$$\Rightarrow \quad \ln y = (\ln x). \ln(1 + e^x) ,$$

Then by differentiating both sides of this equation with respect to x, we have:

$$\Rightarrow \quad \frac{d}{dx}\ln y = \frac{d}{dx}[(\ln x). \ln(1 + e^x)]$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \ln(1+e^x).\frac{d}{dx}(\ln x) + (\ln x)\frac{d}{dx}\ln(1+e^x)$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \ln(1+e^x).(\frac{1}{x}) + (\ln x).\frac{1}{(1+e^x)}.\frac{d}{dx}(1+e^x)$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\ln(1+e^x) + (\ln x).\frac{1}{(1+e^x)}.(e^x)$$

$$\Rightarrow \quad \frac{dy}{dx} = y\{\frac{1}{x}\ln(1+e^x) + \frac{(\ln x).e^x}{(1+e^x)}\},$$

And finally $$\frac{dy}{dx} = (1+e^x)^{\ln x}.\{\frac{1}{x}\ln(1+e^x) + \frac{(\ln x).e^x}{1+e^x}\}.$$

Example (3) Use the logarithmic differentiation to find y' for the following function:

$$y = \frac{\sqrt{1-x^2}}{1-2x}$$

Solution

By taking the natural logarithm of both sides of the equation, then we obtain

$$\ln y = \ln\frac{\sqrt{1-x^2}}{1-2x}$$

$$\Rightarrow \quad \ln y = \ln\sqrt{1-x^2} - \ln(1-2x)$$

$$\Rightarrow \quad \ln y = \ln(1-x^2)^{\frac{1}{2}} - \ln(1-2x)$$

$$\Rightarrow \quad \ln y = \frac{1}{2}.\ln(1-x^2) - \ln(1-2x),$$

Now, by differentiating both sides of this equation with respect to x, we have:

$$\Rightarrow \quad \frac{d}{dx}\ln y = \frac{d}{dx}[\frac{1}{2}.\ln(1-x^2) - \ln(1-2x)]$$

$$\Rightarrow \quad \frac{d}{dx}\ln y = \frac{1}{2}\frac{d}{dx}[\ln(1-x^2) - \ln(1-2x)]$$

$$\Rightarrow \quad \frac{d}{dx}\ln y = \frac{1}{2}\{\frac{d}{dx}\ln(1-x^2) - \frac{d}{dx}\ln(1-2x)\}$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\{\frac{1}{(1-x^2)}\frac{d}{dx}(1-x^2) - \frac{1}{(1-2x)}\frac{d}{dx}(1-2x)\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\{\frac{1}{(1-x^2)}.(-2x)-\frac{1}{(1-2x)}.(-2)\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=\{\frac{-x}{(1-x^2)}.-\frac{-1}{(1-2x)}\}$$

$$\Rightarrow \frac{dy}{dx}=y\{\frac{1}{(1-2x)}-\frac{x}{(1-x^2)}\},$$

And finally $$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{1-x^2}}{1-2x}\{\frac{1}{1-2x}-\frac{x}{1-x^2}\}.$$

Example (4) Find the derivative of $y=2e^x x^{4x}$

Solution

This function has the form $u(x)^{v(x)}$, hence we may use the technique of logarithmic differentiation. By taking the natural logarithm of both sides of the equation, we obtain:

$$\ln y=\ln 2e^x x^{4x}$$

$$\Rightarrow \quad \ln y=\ln 2+\ln e^x+\ln x^{4x}$$

$$\Rightarrow \quad \ln y=\ln 2+x+4x.\ln x$$

Now, by differentiating both sides of this equation with respect to x, we have:

$$\frac{d}{dx}\ln y=\frac{d}{dx}(\ln 2+x+4x.\ln x)$$

$$\Rightarrow \frac{d}{dx}\ln y=\frac{d}{dx}\ln 2+\frac{d}{dx}x+\frac{d}{dx}[4x.\ln x]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=0+1+\ln x.\frac{d}{dx}(4x)+4x.\frac{d}{dx}(\ln x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=1+\ln x.(4)+4x.\frac{1}{x}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=1+4\ln x+4$$

$$\Rightarrow \frac{dy}{dx}=y(5+4\ln x)$$

$$\Rightarrow \frac{dy}{dx}=2e^x x^{4x}(5+4\ln x).$$

Example (5) If $y = (\ln x)^{\ln x}$, find $\frac{dy}{dx}$ when x = e.

Solution

By taking the natural logarithm of both sides of the equation, we obtain:

$$\ln y = \ln(\ln x)^{\ln x}$$

$$\Rightarrow \quad \ln y = (\ln x).\ln(\ln x),$$

Thus, by differentiating both sides of this equation with respect to x, we obtain:

$$\frac{d}{dx}\ln y = \frac{d}{dx}[\ln x.\ln(\ln x)]$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \ln(\ln x).\frac{d}{dx}(\ln x) + \ln x.\frac{d}{dx}[\ln(\ln x)]$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \ln(\ln x).(\frac{1}{x}) + \ln x.(\frac{1}{\ln x})\frac{d}{dx}(\ln x)$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\ln(\ln x) + \ln x.(\frac{1}{\ln x}).(\frac{1}{x})$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\ln(\ln x) + (\frac{1}{x}) \Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{x}[\ln(\ln x) + 1]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}[\ln(\ln x) + 1] \quad \Rightarrow \frac{dy}{dx} = \frac{(\ln x)^{\ln x}}{x}[\ln(\ln x) + 1]$$

Now as x = e then the derivative is $\left.\frac{dy}{dx}\right|_{x=e} = \frac{(\ln e)^{\ln e}}{e}.[\ln(\ln e) + 1] = \frac{1}{e}[\ln(1) + 1] = \frac{1}{e}[0 + 1] = \frac{1}{e}$.

Home work: Solve the book pages 617 and 618, the following problems:

[8] Use the logarithmic differentiation to find the derivative of the function

$y = \sqrt{\dfrac{x^2+5}{x+9}}$.

[8] Use the logarithmic differentiation to find the derivative of the function

$y = \sqrt[3]{\dfrac{6(x^3+1)^2}{x^6 e^{-4x}}}$.

[19] Find y' if $y = 4e^x x^{3x}$.

[21] If $y = (4x-3)^{2x+1}$, then find $\dfrac{dy}{dx}$ when x = 1.

[26] If $y = x^x$, find the relative rate of change of y with respect to x when x = 1.