Section P.1 the Real numbers

Set: a set is a collection of well defined elements that have a common property.

The set A = {1, 2, 3, 5} is finite set because it has a finite number of elements, and also the set B = {0, 2, 4, 5, 6} is finite set. The element 1 is belonging to the set A, then we write $1 \in A$. However the set of all **Natural numbers** N = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, … } is infinite set.

We say that set A is a subset of set B, if every element of the set A is also an element of the set B. We write it as $A \subseteq B$ or $A \subset B$. For instance $A \subset N$ but $B \not\subset N$ because $0 \notin N$.

Set-builder notation:

When we write infinite sets it is preferable to use this type of writing sets, for instance the set of **even numbers** {0, 2, 4, 6, 8, 10, … } can be rewritten in the set-builder notation as follows:

$$\{x \mid x \text{ is even number}\},$$

and the set of **odd numbers** {1, 3, 5, 7, 9, 11 } can be rewritten in the set-builder notation as follows:

$$\{x \mid x \text{ is odd number}\},$$

Union of sets:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\},$$

for instance, $A \cup B$ = {0, 1, 2, 3, 4, 5, 6}.

Intersection of sets:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\},$$

for instance, $A \cap B$ = {2, 5}.

Difference of sets:

$$A - B = \{x \mid x \in A \text{ and } x \notin B\},$$

for instance, A−B = {1, 3}, and B−A = {0, 4, 6}.

Remark: the symbol Ø is used to represent the empty set.

Sets of numbers:

We have already introduced the set of Natural numbers N, then we define the set of **whole numbers** W as $W = \{0,1,2,3,4,5,6,...\}$.

It is clear that $N \subset W$ and $W = \{0\} \cup N$.

The set of **integers** is denoted by the letter Z, and defined as follows:

$$Z = \{..., -5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5\}.$$

The set of **rational numbers** Q all numbers that can be written on the form $\frac{a}{b}$ (where, both a and b are integers and $b \neq 0$). i.e,

$$Q = \{\text{the set of all terminating or repeating decimals}\}.$$

For instance, $0.8 \in Q,\ -0.65 \in Q,\ 0.0002 \in Q,\ 2 \in Q\ (because\ 2 = \frac{2}{1} = 2.0),\ -3 = -3.0 \in Q, ...$

We write Q in the set-builder notation as follows:

$$Q = \{\frac{a}{b} \middle| a \in Z, b \in Z \ \ and \ \ b \neq 0\}.$$

The **periodic number** is a number where its decimal digits are repeating, for instance the number 0.121212121212…, has two repeating decimals 1 and 2, and therefore it is rewritten as $0.\overline{12}$. Also, 0.95353535353… is rewritten as $0.9\overline{53}$. Thus all periodic numbers are belonging to Q. i.e. $0.\overline{12} \in Q$ and $0.9\overline{53} \in Q$.

On the other hand the numbers $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{10}, \sqrt{11}, \sqrt{12}, ...$ has nonterminating decimals, so we define the set of **irrational numbers** Q' as follows:

$$Q' = \{\text{all nonterminating, nonrepeating decimals}\}.$$

Now, $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{10}, \sqrt{11}, \sqrt{12}, ...$ are all belonging to Q', but

$\sqrt{4} \notin Q',\ \sqrt{9} \notin Q',\ \sqrt{16} \notin Q', \sqrt{25} \notin Q', \sqrt{36} \notin Q',\ \sqrt{49} \notin Q', \sqrt{64} \notin Q', \sqrt{81} \notin Q', \sqrt{100} \notin Q', ..$

The set of **all real numbers** contains the sets of all rational or irrational numbers and denoted by R. Thus,

$$R = Q \cup Q'.$$

$\therefore 1 \in R,\ 0 \in R, -1 \in R,\ 0.5 \in R, 0.\overline{13} \in R, \sqrt{3} \in R.$

The relation between the all mentioned sets of numbers is given as follows:

$$N \subset W \subset Z \subset Q \subset R.$$

The prime number:

It is the positive number that has only two factors 1 and itself. The numbers 2, 3, 5, 7, 11, 13, … are all prime (because each one of them has only two factors).

Notice that 1 is not prime number because it has only one factor.

Example (1) determine whether each number is an integer, a rational number, an irrational number, a real number or a prime number:

$$-\frac{1}{5},\ 0,\ -44,\ \pi,\ 3.14,\ 5.05005000500005...,\ \sqrt{81},\ 53.$$

***Solution*:**

Integers: $-44,\ 0,\ \sqrt{81},\ 53.$

Rational numbers: $-\frac{1}{5},\ 0,\ -44,\ 3.14,\ \sqrt{81},\ 53.$

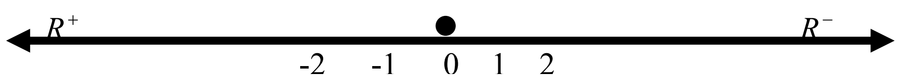
Irrational numbers: $\pi,\ 5.05005000500005....$

Real numbers: $-\frac{1}{5},\ 0,\ -44,\ \pi,\ 3.14,\ 5.05005000500005...,\ \sqrt{81},\ 53.$

Prime numbers: 53.

Absolute value and distance:

All real numbers are represented on the number line as follows:



Where, R^+ is the set of all positive real numbers, and R^- is the set of all negative real numbers. The point corresponding to zero on number line is called the **origin**.

$|2|$ means the distance is 2 units between the origin and a point on the right of this origin.

$|-2|$ means the distance is 2 units between the origin and a point on the left of this origin.

i.e. $|2| = |-2| = 2$ units.

Now, $|a|$ is called the absolute value of the real number a and is defined as

$$|a| = \begin{cases} a & if \quad a \geq 0, \\ -a & if \quad a < 0. \end{cases}$$

If a > 0, then $|a| = |-a| = a$.

Example (2) write each expression without absolute value symbol

$(i)\ -|-5|,\ (ii)\ |3|.|-4|,\ (iii)\ |\pi^2+10|,\quad (iv)\ |x+6|+|x-2|,\quad given\ \ 2<x<3,$

$(v)\ |x+1|+|x-3|,\quad given\ \ x>3.$

Solution:

(i) $-|-5| = -(5) = -5.$

(ii) $|3|.|-4| = (3).(4) = 12.$

(iii) $|\pi^2+10| = \pi^2+10.$

(iv) $\because 2<x<3 \Rightarrow x-2>0\ \ and\ \ also\, x+6>0,$

$\therefore |x-2| = x-2\ \ and\ \ |x+6| = x+6.$

Now, $|x+6|+|x-2| = (x+6)+(x-2) = 2x+4.$

(v) $\because x>3 \Rightarrow x+1>0\ \ and\ \ also\ \ x-3>0,$

$\therefore |x+1| = x+1\ \ and\ \ |x-3| = x-3.$

Now, $|x+1|+|x-3| = (x+1)+(x-3) = 2x-2.$

The distance:

The distance between any two points a, b on the number line is defined as follows:

$$d(a,b) = |a-b|.$$

Example (3) use the absolute value notation to describe the distance between:

(i) x and 3 (ii) a and −2.

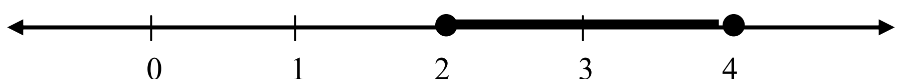
Solution:

(i) $d(x,3) = |x-3|.$

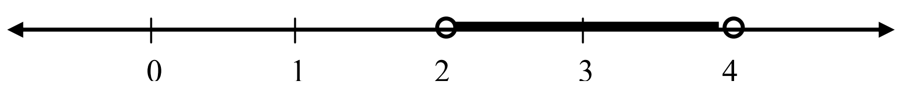
(ii) $d(a,-2) = |a-(-2)| = |a+2|.$

Interval notation

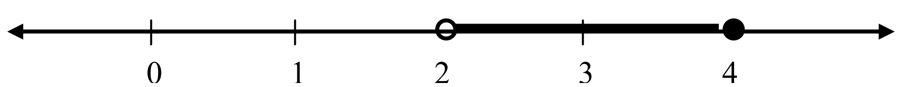
As we have already known that, the set of all real numbers R is represented by the number line. An interval is a subset of R, for example the subset of real numbers $\{x| \, 2 \le x \le 4\}$ is represented by the closed interval [2, 4], and is graphed on the number line as:



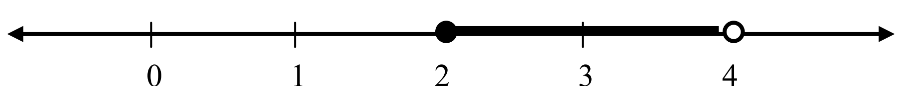
The subset of real numbers $\{x| \, 2 < x < 4\}$ is represented by the open interval (2, 4), and is graphed on the number line as:



The subset of real numbers $\{x| \, 2 < x \le 4\}$ is represented by the half-open interval (2, 4], and is graphed on the number line as:



The subset of real numbers $\{x| \, 2 \le x < 4\}$ is represented by the half-open interval [2, 4), and is graphed on the number line as:

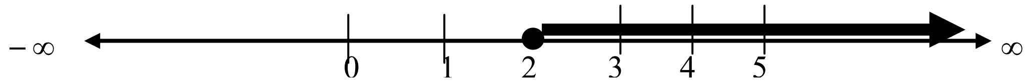


Definition:

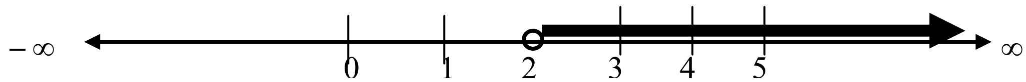
The number ∞ is bigger than any existed real number. The number $-\infty$ is smaller than any existed real number.

$\therefore \; \infty \notin R$ and $-\infty \notin R$.

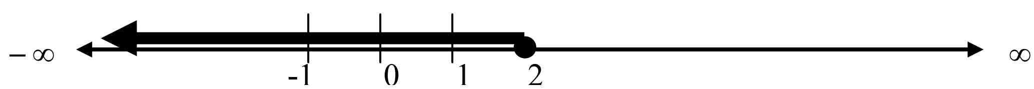
The subset of real numbers $\{x|\ x \geq 2\}$ equal the interval $[2, \infty)$ and is represented on the number line by the graph:



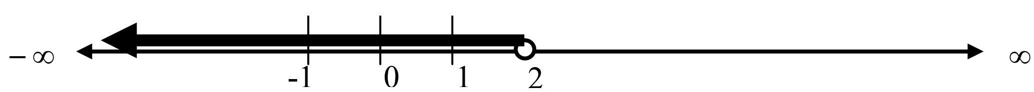
The subset of real numbers $\{x|\ x > 2\}$ equal the interval $(2, \infty)$ and is represented on the number line by the graph:



The subset of real numbers $\{x|\ x \leq 2\}$ equal the interval $(-\infty,\ 2]$ and is represented on the number line by the graph:



The subset of real numbers $\{x|\ x < 2\}$ equal the interval $(-\infty,\ 2)$ and is represented on the number line by the graph:

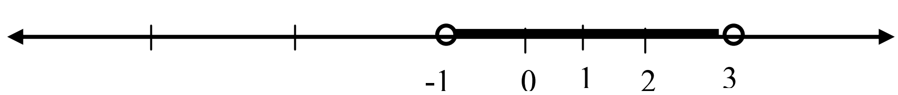


Example (4a) write the following intervals in set-builder notation and graph them:

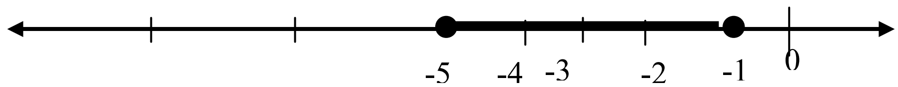
(i) $(-2, 3)$, (ii) $[-5, -1]$, (iii) $[-1, \infty)$.

Solution:

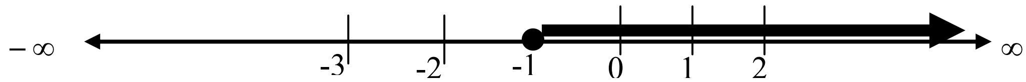
(i) $(-2, 3) = \{x|\ -2 < x < 3\}$, and its graph on number line is:



(ii) $[-5,-1]=\{x|-5\le x\le -1\}$, and its graph on number line is:



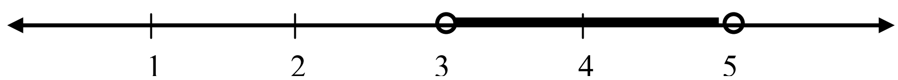
(iii) $[-1,\infty)=\{x|-1\le x\}$, and its graph on number line is:



Example (4b) write the interval that equal to the following subset of real numbers, $\{x|\ 3<x<5\}$, then graph it on number line.

Solution:

$\{x|\ 3<x<5\}=(3,5)$, and its graph on number line is:



Order operations

The quantities, $5x$, $16y$, $14xy$, $25x^2y^3$ are called algebraic terms, for instance the algebraic term 5x has a variable x and numerical coefficient 5.

The algebraic expression consists of a collection of algebraic terms.

The order of operations:

(1) evaluate exponential expressions,

(2) $\times$ and $\div$ from left to right,

(3) + and − from left to right.

To evaluate a variable expression, replace the variables by their given values and then use the order of operations.

Example (6) evaluate the variable expression for the following:

$(i)\ -2x^2y^2, \quad for\ \ x=3,\ \ y=-2.$

$(ii)\ 2xyz, \quad for\ \ x=3,\ \ y=-2\ \ and\ \ z=-1.$

$(iii)\ \dfrac{x^2+y^2}{x+y}, \quad for\ \ x=1,\ \ y=2.$

$(iv)\ \ xy-z(x-y)^2, \quad for\ \ x=3,\ \ y=-2\ \ and\ \ z=-1.$

Solution:

$(i)\ -2x^2y=-2(3)^2(-2)^2=-2(9)(4)=-2(36)=-72.$

(ii) $2xyz=2(3)(-2)(-1)=2(3)(2)=2(6)=12.$

$(iii)\ \dfrac{x^2+y^2}{x+y}=\dfrac{(1)^2+(2)^2}{1+2}=\dfrac{1+4}{3}=\dfrac{5}{3}.$

$$
\begin{aligned}
(iv)\ \ xy-z(x-y)^2 &= 3(-2)-(-1)(3-(-2))^2 \\
&= -6+(1)(3+2)^2 \\
&= -6+(1)(5)^2 \\
&= -6+25 \\
&= 19.
\end{aligned}
$$

Properities of real numbers

Let a, b and c be real numbers, then they satisfy the following properities:

Closure	a + b is a unique real number	ab is a unique real number
Commutative	a + b = b + a	ab = ba
Associative	(a + b) + c = a + (b + c)	(ab)c = a(bc)
Identity	There exists a unique real number 0 such that: a + 0 = 0 + a = a.	There exists a unique real number 1 such that: a(1) = (1)a = a
Inverse	For each $a \in R$ there exists $-a \in R$ such that: a + (-a) = (-a) + a = 0	For each $a \in R$ there exists $\frac{1}{a} \in R$ such that: $a(\frac{1}{a}) = (\frac{1}{a})a = 1$
Distributive	a(b + c) = ab + ac.	

Example (7) simplify the variable expressions of the following:

$(i)\ 3(2+x),\quad (ii)\ \frac{2}{3}a+\frac{5}{6}a,\quad (iii)\ 2+3(2x-5),\quad (iv)\ 5-3(4x-2y),$
$(v)\ 3(2a-4b)-4(a-3b),\quad (vi)\ 5a-2[3-2(4a+3)].$

Solution

$(i)\ 3(2+x)=3(2)+3(x)=6+3x.$

$(ii)\ \frac{2}{3}a+\frac{5}{6}a=(\frac{2}{3}+\frac{5}{6})a=(\frac{4}{6}+\frac{5}{6})a=(\frac{9}{6})a=\frac{3}{2}a.$

$(iii)\ 2+3(2x-5)=2+6x-15=-13+6x.$

$(iv)\ 5-3(4x-2y)=5-3(4x)-3(-2y)=5-12x+6y.$

$(v)\ 3(2a-4b)-4(a-3b)=6a-12b-4a+12b=(6-4)a+(12-12)b=2a+(0)b=2a.$

$(vi)\ 5a-2[3-2(4a+3)]=5a-6+4(4a+3)=5a-6+16a+12=(5+16)a+(12-6)=21a+6.$