Example (1) Find $\frac{d}{dx}(c)$, where c is constant.

***Solution*:**

Since $f(x)=c$, then $f(x+h)=c$

$$\therefore \frac{f(x+h)-f(x)}{h}=\frac{c-c}{h}=\frac{0}{h}=0,$$

$$\therefore \frac{d}{dx}(c)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{0}{h}=\lim_{h\to 0}0=0.$$

Now, we have the following rule:

Rule 1: the derivative of a constant function is zero.

Example 2 find $\frac{d}{dx}x^3$ using the definition

***Solution*:**

Since $f(x)=x^3$, then

$$f(x+h)=(x+h)^3=x^3+3x^2h+3xh^2+h^3,$$

$$\therefore \frac{f(x+h)-f(x)}{h}=\frac{x^3+3x^2h+3xh^2+h^3-(x^3)}{h}$$

$$=\frac{+3x^2h+3xh^2+h^3}{h}=\frac{h(3x^2+3xh+h^2)}{h}=3x^2+3xh+h^2.$$

$$\therefore f'=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}(3x^2+3xh+h^2)=(3x^2+0+0)=3x^2.$$

For general x^n we have the following rule:

Rule 2: $\frac{d}{dx}x^n=nx^{n-1}$

Example (3) find the derivative of the following functions:

$(i)\ f(x)=x^4,\quad (ii)\ f(x)=x,\quad (iii)\ f(x)=x\sqrt{x},\quad (iv)\ f(x)=\sqrt{x}.$

Solution:

(i) $\frac{dx^4}{dx} = 4x^{4-1} = 4x^3$,

(ii) $\frac{d(x)}{dx} = (1)x^{1-1} = (1)x^0 = (1).(1) = 1$,

(iii) $\frac{d(x\sqrt{x})}{dx} = \frac{d(x^{3/2})}{dx} = (3/2)x^{(3/2)-1} = (3/2)x^{1/2} = (3/2)\sqrt{x}$,

(iv) $\frac{d\sqrt{x}}{dx} = \frac{d(x^{1/2})}{dx} = (1/2)x^{(1/2)-1} = \frac{1}{2}x^{(-1/2)} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$.

Rule 3: $\frac{d}{dx}(c.f(x)) = c.f'(x)$

Proof:

Let $g(x) = c.f(x)$, then $g(x+h) = c.f(x+h)$

$$\therefore \frac{g(x+h)-g(x)}{h} = \frac{c.f(x+h)-c.f(x)}{h} = \frac{c.(f(x+h)-f(x))}{h},$$

$$\therefore \frac{d}{dx}(c.f(x)) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{c.(f(x+h)-f(x))}{h} = c.\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = c.\frac{d}{dx}f(x).$$

Example 4 differentiate the following functions:

(i) $f(x) = 3x^6$, (ii) $f(t) = -\frac{1}{8}t^8$.

Solution:

(i) $\frac{df(x)}{dx} = \frac{d(3x^6)}{dx} = 3.\frac{dx^6}{dx} = (3).(6)x^5 = 18x^5$.

(ii) $\frac{df(t)}{dt} = \frac{d(-\frac{1}{8}t^8)}{dt} = -\frac{1}{8}.\frac{dt^8}{dt} = (-\frac{1}{8}).(8)t^{8-1} = -t^7$.

Rule 4 derivative of sum of two functions equals sum of their derivatives

$$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x),$$

derivative of difference of two functions equals difference of their derivatives

$$\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x).$$

Example (5) find the derivative of the following functions:

$(i)\ f(x)=x+3,\quad (ii)\ f(x)=4x^2-2x+3,\quad (iii)\ f(x)=-x^8+x^5,$

$(iv)\ f(x)=x(3x^2-10x+7),\quad (v)\ f(x)=\dfrac{5x^4+x^2}{12\sqrt{x}}.$

Solution

(i) $f'(x)=\dfrac{df(x)}{dx}=\dfrac{d(x+3)}{dx}=\dfrac{d(x)}{dx}+\dfrac{d(3)}{dx}=1+0=1.$

(ii) $$f'(x)=\frac{df(x)}{dx}=\frac{d(4x^2-2x+3)}{dx}=\frac{d(4x^2)}{dx}+\frac{d(-2x)}{dx}+\frac{d(3)}{dx}=4\frac{d(x^2)}{dx}-2\frac{d(x)}{dx}+0$$
$$=(4)(2)x-2(1)+0=8x-2.$$

(iii) $f'(x)=\dfrac{df(x)}{dx}=\dfrac{d(-x^8+x^5)}{dx}=\dfrac{d(-x^8)}{dx}+\dfrac{d(x^5)}{dx}=-\dfrac{d(x^8)}{dx}+\dfrac{d(x^5)}{dx}=-8x^7+5x^4.$

(iv) $f(x)=x(3x^2-10x+7)=3x^3-10x^2+7x,$

$$f'(x)=\frac{df(x)}{dx}=\frac{d(3x^3-10x^2+7x)}{dx}=\frac{d(3x^3)}{dx}+\frac{d(-10x^2)}{dx}+\frac{d(7x)}{dx}=3\frac{d(x^3)}{dx}-10\frac{d(x^2)}{dx}+7\frac{d(x)}{dx}$$
$$=(3)(3)x^2-10(2)x+7(1)=9x^2-20x+7.$$

(v)
$$f(x)=\frac{5x^4+x^2}{12\sqrt{x}}=\frac{5x^4+x^2}{12x^{(1/2)}}=\frac{1}{12}x^{-(1/2)}(5x^4+x^2)=\frac{1}{12}(5x^{4-(1/2)}+x^{2-(1/2)})$$
$$=\frac{1}{12}(5x^{(7/2)}+x^{(3/2)}),$$

$$f'(x)=\frac{df(x)}{dx}=\frac{d(\frac{5}{12}x^{(7/2)}+\frac{1}{12}x^{(3/2)})}{dx}=\frac{d(\frac{5}{12}x^{(7/2)})}{dx}+\frac{d(\frac{1}{12}x^{(3/2)})}{dx}=\frac{5}{12}\frac{d(x^{(7/2)})}{dx}+\frac{1}{12}\frac{d(x^{(3/2)})}{dx}$$

$$=(\frac{5}{12})(\frac{7}{2})x^{(7/2)-1}+(\frac{1}{12})(\frac{3}{2})x^{(3/2)-1}=\frac{35}{24}x^{(5/2)}+\frac{1}{8}x^{(1/2)}=\frac{35}{24}\sqrt{x^5}+\frac{1}{8}\sqrt{x}.$$

Example (6) find the slopes of the curve $y=3x^2+4x-8,$ at the points $(0,-8),\ (2,12),\ (-3,7)$.

Solution

The derivative of this function is $\frac{dy}{dx}=y'=6x+4,$

Since $\left.\frac{dy}{dx}\right|_x$ is the slope of the tangent line to the curve of y = f(x) at the point x,

Then, the slope at the point $(0,-8)$ is:

$$\left.\frac{dy}{dx}\right|_{x=0}=6(0)+4=0+4=4.$$

The slope at the point $(2,12)$ is:

$$\left.\frac{dy}{dx}\right|_{x=2}=6(2)+4=12+4=16.$$

The slope at the point $(-3,7)$ is:

$$\left.\frac{dy}{dx}\right|_{x=-3}=6(-3)+4=-18+4=-14.$$

Example 7 find all points on the curve $y=\frac{x^3}{3}-x+1,$ where the tangent line is horizontal.

Solution

The derivative of this function is $\frac{dy}{dx}=3(\frac{1}{3})x^2-1=x^2-1,$

If the tangent is horizontal then its slope equals zero

But, know that $\left.\frac{dy}{dx}\right|_x$ is the slope of the tangent line to the curve of y = f(x) at the point x, then to obtain all the points of the curve of y at which the tangent line is horizontal put $\frac{dy}{dx}=0$, this implies that $x^2-1=0$. The solutions of the last equation are $x=-1$ or $x=1$.

At $x=-1$, then $y=\frac{(-1)^3}{3}-(-1)+1=-\frac{1}{3}+1+1=-\frac{1}{3}+2=\frac{5}{6}$, i.e the tangent line to the curve of y is horizontal to the point $(-1,\frac{5}{6})$ which lies on this curve.

Similarly, at $x=1$, then $y=\frac{(1)^3}{3}-(1)+1=\frac{1}{3}$, i.e the tangent line to the curve of y is horizontal to the point $(1,\frac{1}{3})$ which lies also on this curve.

Home work: solve book pages 551 and 552, the following problems:

Differentiate the following functions:

[22] $y=-8x^4+\ln 2$,

[34] $f(x)=2x^{(-14/5)}$,

[61] $y=x^2\sqrt{x}$,

[73] $w(x)=\frac{x^2+x^3}{x^2}$,

[78] find all the slopes of the function $y=3x-4\sqrt{x}$ when x = 4, x = 9, x = 25.

[85] find all points on the curve $y=\frac{5}{2}x^2-x^3$, where the tangent line is horizontal.