A Beautiful ‘unsolvable’ Equation

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Abstract

In the movie *A Beautiful Mind*, based in part on the life of John Nash was featured an unsolvable set of equations. It is here demonstrated why these sets of equations were not solvable using known techniques.

1 Introduction

The equations written in the film by Rustle Crowe were the following:

$$V = \{F : R^3 | X \mapsto R^3 \ so \ \nabla \times F \equiv 0\} \tag{1}$$

$$W = \{F = \nabla g\} \tag{2}$$

$$div(V/W) = ? \tag{3}$$

While looking very simple with equation (1) being the initial statement. Equation (2) is a second function, combing the two functions as a product of ∇ thus resulting in an answer (equation 3) should be a matter of simple arithmetic. Except for the fact that this form of mathematics is in fact more complicated and comes by the name of multi-variable calculus. The problem is that equation (3) has no solvable solution in the known number systems. In the next sections we shall examine the difficulty that these three statements present.

2 The meaning of the function V

What is the meaning behind V? First equation (1) states that F is a real three-dimensional function $F : R^3$. It further states that an arbitrary set of coordinates X is to be mapped (placed on to) a real three-dimensional space R^3. Such that the function F satisfies the curl $\nabla \times F \equiv 0$ now this is where things begin to take a turn for the worse, the last statement can be written as:

$$0 = \nabla \times F = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{j} + \left(\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x}\right)\hat{k} \quad (4)$$

This is the definition of a partial differential vector field in three-space. The meaning of F can be prescribed as

$$F' = \frac{dF}{dX} = f'(x, y, z) \quad (5)$$

or how F varies as a function of the derivative of the coordinates x, y, z. To derive F one needs to perform the operation

$$f'(x, y, z) = \lim_{\Delta X \to 0} \frac{\Delta F(x, y, z)}{\Delta X} \quad (6)$$

If $F = 1$ the function would be a straight line however, it would be need to have the vector arrangement of $F = (1, 0, 0)$, in this case $d = 1$, however the function need not be limited to this linear structure. F prime, f' can not take the value 0, if it did as X approached 0 the result would be a singularity and hence no derivative (meaning any solution would have to have a non zero solution). The trick to solving equation (4) then becomes to find the differential values needed to return a sum of zero using real numbers for F. But since $X \mapsto R^3$ there becomes no real clear way to perform this task.

3 The meaning of the function W

Equation (2) then goes on to state that $F = \nabla g$, or that F is the gradient of another function named g. The new function g has then the following definitions:

$$\nabla g = \frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial g}{\partial z}\hat{k} \quad (7)$$

and as to be expected it makes equation (4) that much more difficult to solve. The real kicker is that this is equal to F, in other words the solution to this problem must be a real number. In shorthand the expression would roughly correspond to

$$\nabla \times F = \left(\frac{\partial \left(\frac{\partial g}{\partial x}\hat{\imath} + \frac{\partial g}{\partial y}\hat{\jmath} + \frac{\partial g}{\partial z}\hat{k} \right)_z}{\partial y} - \frac{\partial \left(\frac{\partial g}{\partial x}\hat{\imath} + \dots \right)_y}{\partial z} \right) \hat{\imath} + \dots \tag{8}$$

3.1 The meaning of (V/W)

The operation of (V/W) seems simple enough, but all that equation (2) does is to state that $F = \nabla g$. As such we can simply ignore the right hand side so that the operation becomes

$$\emptyset = \frac{\nabla \times F}{F} = \frac{0}{F} \tag{9}$$

This operation then assumes that $F = I$, since the function F is an identity and hence must be a linear orthogonal function (more precisely $F = (1, 1, 1)$).

4 Equation 3 the divergence equation

Lastly there is the final equation (3), which is a divergence equation, i.e. it takes the form:

$$\nabla \cdot (V/W) = \frac{\partial (0/F)_x}{\partial x} + \frac{\partial (0/F)_y}{\partial y} + \frac{\partial (0/F)_z}{\partial z} \tag{10}$$

the final equation no matter how ugly does have a solution. However there is no known solution that satisfies the criteria set aside by equation (1). The best solution to equation (3) would be $\emptyset$. The problem is unsolvable because an empty set of vector fields do not diverge. It can not be said that a vector field does not exist because of equation (1). It is much the same phenomenon one encounters in gravitation as a body falls into the singularity of a black hole, the number system fails. In this since it is the abyss of the number system, if you look into the problem all that it does is to look back at you.