1

(i) Let $P = (a, 0)$, $Q = (0, b)$, the equation of L is

$$\frac{x}{a} + \frac{y}{b} = 1$$

it passes A(4, 3)

$$\frac{4}{a} + \frac{3}{b} = 1$$

area of $\Delta POQ = 24.5$

$$\frac{ab}{2} = 24.5$$

solving

$$\begin{cases} \frac{4}{a} + \frac{3}{b} = 1 \\ \frac{ab}{2} = 24.5 \end{cases}$$

$$(a, b) = (7, 7) \quad \text{or} \quad \left(\frac{28}{3}, \frac{21}{4}\right)$$

∴ equations of L are

$$x + y = \underline{\underline{7}}$$

and

$$9x + 16y = \underline{\underline{84}}$$

(ii) Equation L_1 and L_2 passing through A(4, 3) are in the form

$$y - 3 = m(x - 4), \text{ where } m \text{ is the slope}$$

$$\text{or } mx - y + 3 - 4m = 0$$

Being $\sqrt{5}$ units from origin

$$\left| \frac{m(0) - (0) + 3 - 4m}{\sqrt{m^2 + 1}} \right| = \sqrt{5}$$

$$(3 - 4m)^2 = 5(m^2 + 1)$$

$$11m^2 - 24m + 4 = 0$$

$$(11m - 2)(m - 2) = 0$$

$$m = \frac{2}{11} \quad \text{or} \quad 2$$

hence L_1: $2x - y - 5 = 0$

and L_2: $2x - 11y + 25 = 0$

(iii) slope of L_1, $m_1 = 2$

slope of L_2, $m_2 = \frac{2}{11}$

$$\tan\alpha = \frac{2 - \frac{2}{11}}{1 + 2 \times \frac{2}{11}} = \frac{\frac{20}{11}}{\frac{15}{11}} = \frac{4}{3}$$

$$\alpha = \tan^{-1}\frac{4}{3}$$

$$= \underline{\underline{53°8'}}$$

2

(a) Equation of L is given by

$$59x - 76y + 46 + \lambda(44x - 27y + 57) = 0$$

$$\text{or} \quad (59 + 44\lambda)x - (76 + 27\lambda)y + 46 + 57\lambda = 0$$

$$\frac{x}{-\frac{46 + 57\lambda}{59 + 44\lambda}} + \frac{y}{\frac{46 + 57\lambda}{76 + 27\lambda}} = 1$$

$$\text{y-intercept of L} = 1 = \frac{46 + 57\lambda}{76 + 27\lambda}$$

$$\therefore \lambda = 1$$

Equation of L: $103x - 103y + 103 = 0$

or $x - y + 1 = 0$

(b) distance from the origin to L

$$d = \left| \frac{(0) - (0) + 1}{\sqrt{1^2 + 1^2}} \right| = \underline{\underline{\frac{1}{\sqrt{2}} \text{ unit.}}}$$

3

(a) area of triangle with vertices $A(-1, 2)$, $B(a - 3, a + 1)$, $C(a, a + 2)$

$$\Delta = \frac{1}{2}\{(-1)(a + 1 - a - 2) + (a - 3)(a + 2 - 2) + a(2 - a - 1)\}$$

$$= \frac{1}{2}\{1 + a^2 - 3a + a - a^2\}$$

$$= \underline{\underline{\frac{1}{2}(1 - 2a) \text{ sq. units}}}$$

(b) If A, B, C are collinear, the $\Delta = \frac{1}{2}(1 - 2a) = 0$

$$\therefore a = \underline{\underline{\frac{1}{2}}}$$

(c) Equation of BC:

$$\frac{y - a - 1}{x - a + 3} = \frac{a + 2 - a - 1}{a - a + 3}$$

$$x - 3y + 2a + 6 = 0$$

(d) $\frac{x}{-2a - 6} + \frac{y}{\frac{2a + 6}{3}} = 1$ is the intercepts form of BC

$$\therefore P = (-2a - 6, 0), \quad Q = \left(0, \frac{2a + 6}{3}\right)$$

(e) area of ΔOPQ is unity

$$\frac{1}{2}(-2a - 6)\left(\frac{2a + 6}{3}\right) = \pm 1$$

$$(2a + 6)^2 = \pm 3$$

Only $(2a + 6)^2 = 3$ is possible

$$\therefore 2a + 6 = \pm\sqrt{3}$$

$$a = \underline{\underline{\frac{-6 \pm \sqrt{3}}{2}}}$$

4

(a) Line L: $(1 + \lambda)x + (3 - \lambda)y = 2(1 + 3\lambda)$

hence $x + \lambda x + 3y - \lambda y - 2 - 6\lambda = 0$

$$x + 3y - 2 + \lambda(x - y - 6) = 0$$

solving

$$\begin{cases} x + 3y - 2 = 0 \\ x - y - 6 = 0 \end{cases}$$

$$x = 5$$

$$y = 1$$

∴ $Q = (5, -1)$ is the fixed point which L must pass through.

(b) If $L /\!/ L_1$: $2x + 3y = 5$,

then

$$\frac{1 + \lambda}{2} = \frac{3 - \lambda}{3}$$

$$\lambda = \frac{3}{5}$$

$$\therefore \text{L:} \quad x + 3y - 2 + \frac{3}{5}(x - y - 6) = 0$$

$$\text{L:} \quad 2x + 3y - y = 0$$

(c) L has x-intercept = 3, hence L passes (3, 0)

$$\therefore (1 + \lambda)(3) = 2(1 + 3\lambda)$$

$$3 + 3\lambda = 2 + 6\lambda$$

$$\lambda = \frac{1}{3}$$

if L passes (0, b)

$$\left(1 + \frac{1}{3}\right)(0) + \left(3 - \frac{1}{3}\right)(b) = 2\left(1 + 3\left(\frac{1}{3}\right)\right)$$

$$b = 4 \div \frac{8}{3}$$

$$= \frac{3}{2}$$

$$\therefore \text{y-intercept of L} = \underline{\underline{\frac{3}{2}}}.$$

5

Any line L passing through the intersection point of

L_1: $x - 2y - 4 = 0$

L_2: $2x + y - 4 = 0$

is given by

L: $x - 2y - 4 + \lambda(2x + y - 4) = 0$

(a) If L has slope 2

$$\therefore -\frac{1 + 2\lambda}{-2 + \lambda} = 2$$

$$1 + 2\lambda - 4 + 2\lambda = 0$$

$$\lambda = \frac{3}{4}$$

$$\therefore \text{L:} \quad 4(x - 2y - 4) + 3(2x + y - 4) = 0$$

$$\text{L:} \quad 10x - 5y - 28 = 0$$

(b) If L passes (3, −2)

$$3 - 2(-2) - 4 + \lambda(2(3) + (-2) - 4) = 0$$

$$-5 + 0 \times \lambda = 0$$

However, rewrite L as

$$2x + y - 4 + k(x - 2y - 4) = 0$$

and substitute (3, −2)

$$2(3) + (-2) - 4 + k[(3) - 2(-2) - 4] = 0$$

$$k = \frac{0}{5} = 0$$

∴ L: $2x + y - 4 = 0$

hence L coincide with L_2.

(c) L is perpendicular to $5x + 3y = 0$

$$\therefore \text{slope of L} = \frac{3}{5} = -\frac{1 + 2\lambda}{-2 + \lambda}$$

$$-6 + 3\lambda + 5 + 10\lambda = 0$$

$$\lambda = \frac{1}{13}$$

∴ L: $15x - 25y - 56 = 0$

6

(a) Equation of line L with slope m and passes A(1, 3)

L: $y - 3 = m(x - 1)$

or $mx - y + 3 - m = 0$

(b) If L is $\sqrt{2}$ units from (2, 6),

$$\pm\sqrt{2} = \frac{m(2) - (6) + 3 - m}{\sqrt{m^2 + 1}}$$

$$2(m^2 + 1) = (m - 3)^2$$

$$m^2 + 6m - 7 = 0$$

$$(m + 7)(m - 1) = 0$$

$$m = \underline{1} \quad \text{or} \quad \underline{-7}$$

∴ the equations of L are L_1: $x - y + 2 = 0$ and L_2: $7x + y - 10 = 0$

(c) Line L': $y - 6 = m(x - 2)$ passes D(2, 6)

if L' and $x - y + 2 = 0$ meet at B

L' and $7x + y - 10 = 0$ meet at C

In view of (a), A(1, 3) is the intersection L_1 and L_2 and D is equidistance from L_1 and L_2.

∴ AD is the perpendicular bisector of isosceles triangle ABC.

$$\text{slope of AD} = \frac{6 - 3}{2 - 1} = 3$$

slope of BC = m

as AD ⊥ BC, $m(3) = -1$

$$m = -\frac{1}{3}$$

hence equation of BC is

$$y - 6 = -\frac{1}{3}(x - 2)$$

or $x + 3y - 20 = 0$

(d) Solving

$$\begin{cases} x + 3y - 20 = 0 \\ x - y + 2 = 0 \end{cases}$$

we get

$$B = \left(\frac{7}{2}, \frac{11}{2}\right)$$

solving

$$\begin{cases} x + 3y - 20 = 0 \\ 7x + y - 10 = 0 \end{cases}$$

we get

$$C = \left(\frac{1}{2}, \frac{13}{2}\right)$$

$$\text{area of } \Delta ABC = \frac{1}{2}\left\{(1)\left(\frac{11}{2} - \frac{13}{2}\right) + \frac{7}{2}\left(\frac{13}{2} - 3\right) + \frac{1}{2}\left(3 - \frac{11}{2}\right)\right\}$$

$$= \frac{1}{2}\left\{-1 + \frac{49}{4} - \frac{5}{4}\right\}$$

$$= \underline{\underline{5 \text{ sq. units.}}}$$

7

(a) L_1: $3x - y + 12 = 0$ and L_1, L_2 pass through A(1, 1)

acute angle between L_1 and L_2 = θ

acute angle between L_2 and L_3 = θ and $\tan\theta = \frac{1}{2}$

Let slope of L_1, L_2 = m, slope of L_3 = 3

$$\therefore \tan\theta = \left| \frac{m - 3}{1 + 3m} \right| = \frac{1}{2}$$

$$\frac{m - 3}{1 + 3m} = \pm\frac{1}{2}$$

$$\therefore 2m - 6 = 1 + 3m \quad \text{or} \quad 2m - 6 = -1 - 3m$$

$$m = -7 \quad \text{or} \quad m = 1$$

∴ slope of L_1 = 1; slope of L_2 = −7

(b) equation of L_1:

$$\frac{y - 1}{x - 1} = 1$$

$$x - y = 0$$

equation of L_2:

$$\frac{y - 1}{x - 1} = -7$$

$$7x + y - 8 = 0$$

(c) Solving $\begin{cases} 3x - y + 12 = 0 \\ x - y = 0 \end{cases}$, $P = (-6, -6)$

Solving $\begin{cases} 3x - y + 12 = 0 \\ 7x + y - 8 = 0 \end{cases}$, $Q = \left(-\frac{2}{5}, \frac{54}{5}\right)$

15. (a) Let the equation of circle be $(x+2)^2+(y-2)^2+A(x+2)+B(y-2)=0$.

As B and C lie on this circle, we have

$$\begin{cases}(2+2)^2+(2-2)^2+A(2+2)+B(2-2)=0 & \quad .....(1)\\ (-2+2)^2+(-4-2)^2+A(-2+2)+B(-4-2)=0 & \quad .....(2)\end{cases}$$

From (1): $16+4A=0$, $A=-4$ From (2): $36-6B=0$, $B=6$

∴ The equation of the circle is

$$(x+2)^2+(y-2)^2-4(x+2)+6(y-2)=0$$
$$x^2+4x+4+y^2-4y+4-4x-8+6y-12=0$$
$$x^2+y^2+2y-12=0$$

(b) Distance from $(-3,0)$ to $x-y-10=0$ is

$$\left|\frac{-3-10}{\sqrt{1^2+1^2}}\right| = \frac{13}{\sqrt{2}}$$

∴ Equation of the circle is $(x+3)^2+y^2=\left(\frac{13}{\sqrt{2}}\right)^2=\frac{169}{2}$

(c) Let the centre of the circle be (a, b)

$$∴ \begin{cases}(a-2)^2+(b-3)^2=(a-2)^2+(b+3)^2 & \quad .....(1)\\ \left(\frac{b+3}{a-2}\right)\left(\frac{2}{3}\right)=-1 & \quad .....(2)\end{cases}$$

From (1):
$$(b-3)^2=(b+3)^2$$
$$(b-3)^2-(b+3)^2=0$$
$$2b(-6)=0$$
$$b=0$$

Put into (2).
$$\left(\frac{3}{a-2}\right)\left(\frac{2}{3}\right)=-1$$
$$2=-(a-2)=-a+2$$
$$a=0$$

∴ The centre is $(0,0)$.

Radius $=\sqrt{(0-2)^2+(0-3)^2}=\sqrt{13}$

The equation of the required circle is $x^2+y^2=13$.

(d) Let the equation of the required circle be
$(x^2+y^2-2x-3y-2)+k(x^2+y^2-3x+y-10)=0$.

As P lies on this circle, we have
$$[(-2)^2+1^2-2(-2)-3(1)-2]+k[(-2)^2+1^2-3(-2)+1-10]=0$$
$$4+2k=0$$
$$k=-2$$

∴ The equation of the required circle is
$$(x^2+y^2-2x-3y-2)-2(x^2+y^2-3x+y-10)=0$$
$$-x^2-y^2+4x-5y+18=0$$
$$x^2+y^2-4x+5y-18=0$$

(e) Let the equation of the required circle be
$$(x^2+y^2+2x-2y+1)+k(x-y+1)=0$$
$$x^2+y^2+(2+k)x-(2+k)y+(1+k)=0$$

Centre of this circle is $\left(\frac{-2-k}{2}, \frac{2+k}{2}\right)$

Hence, $\left(\frac{-2-k}{2}\right)+3\left(\frac{2+k}{2}\right)=3$
$$-2-k+6+3k=6$$
$$-2+2k=0$$
$$k=1$$

∴ The equation of the required circle is $x^2+y^2+3x-3y+2=0$.

(f) $x^2+y^2-4x-2y-11=0$
$$(x-2)^2+(y-1)^2=16$$

∴ The centre of the given circle is $(2,1)$.

Let r be the radius of the required circle.
As the two circles touch each other,

$\sqrt{(2+1)^2+(1-1)^2}=4+r$ or $\sqrt{(2+1)^2+(1-1)^2}=4-r$

$3=4+r$ or $3=4-r$

$r=-1$ (rejected) $r=1$

∴ The equation of the required circle is $(x+1)^2+(y-1)^2=1$.

16. (a) $x^2+y^2+2kx-4ky+6k^2-2=0$
$$(x+k)^2+(y-2k)^2=2-k^2$$

For radius greater than 1, we have
$$\sqrt{2-k^2}>1$$
$$2-k^2>1$$
$$1>k^2$$
$$-1<k<1.$$

(b) Let $P(x, y)$ be a point on the locus.

∴ $x=-k$ and $y=2k$

Hence $\frac{x}{y}=\frac{-1}{2}$ i.e. $2x+y=0$.

∴ The equation of the locus is $2x+y=0$.

17. The equations of C_1 and C_2 can be rewritten as

$\left(x-\frac{7}{2}\right)^2+y^2=\frac{5}{4}$ and $(x-2)^2+(y+3)^2=5$ respectively.

∴ The equation of the line of centre is
$$\frac{y}{x-\frac{7}{2}}=\frac{0-(-3)}{\frac{7}{2}-2}=2$$
$$y=2x-7 \quad .....(1)$$

Equation of the common tangent is
$$(x^2+y^2-7x+11)-(x^2+y^2-4x+6y+8)=0$$
$$-3x-6y+3=0$$
$$x+2y-1=0 \quad .....(2)$$

Put $y=2x-7$ into (2): $x+2(2x-7)-1=0$
$$5x=15$$
$$x=3$$
and $y=2(3)-7=-1$

∴ P is $(3,-1)$.

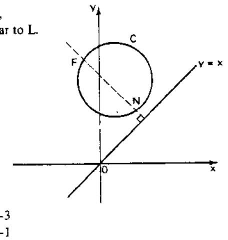
18. (a) Put $y=x$ into the equation of C.
$$(x-2)^2+(x-10)^2=18$$
$$x^2-4x+4+x^2-20x+100=18$$
$$2x^2-24x+86=0$$
$$x^2-12x+43=0 \quad .....(*)$$

Discriminant $=(-12)^2-4(43)$
$=-28<0$.

Hence, the equation (*) does not have real roots and L does not meet the circle C.

(b) As N and F are the nearest and furthest points from L, NF is a diameter of the circle C and NF is perpendicular to L.

Equation of NF is:
$$\frac{y-10}{x-2}=-1$$
$$y-10=-x+2$$
$$y=-x+12$$



Substitute into the equation of C,
$$(x-2)^2+[(-x+12)-10]^2=18$$
$$(x-2)^2+(2-x)^2=18$$
$$2(x-2)^2=18$$
$$(x-2)^2=9$$
$x-2=3$ or $x-2=-3$
$x=5$ or $x=-1$

When $x=5$, $y=7$
When $x=-1$, $y=13$

∴ F is $(-1,13)$ and N is $(5,7)$.

19. (a) (i) The equation of AB is:
$$(x^2+y^2+4x-2y+1)-(x^2+y^2+10x+4y+19)=0$$
$$-6x-6y-18=0$$
$$x+y+3=0$$
$$y=-x-3$$

(ii) As the area of the required circle is minimum, AB is the diameter of the circle.

Put $y=-x-3$ into the equation of C_1,
$$x^2+(-x-3)^2+4x-2(-x-3)+1=0$$
$$x^2+x^2+6x+9+4x+2x+6+1=0$$
$$2x^2+12x+16=0$$
$$x^2+6x+8=0$$
$$(x+2)(x+4)=0$$
$x=-2$ or $x=-4$

When $x=-2$, $y=-(-2)-3=-1$
When $x=-4$, $y=-(-4)-3=1$

∴ A is $(-2,-1)$ and B is $(-4,1)$.

Equation of the required circle is:
$$(x+2)(x+4)+(y+1)(y-1)=0$$
$$x^2+6x+8+y^2-1=0$$
$$x^2+y^2+6x+7=0.$$

(b) Equation of C_1 can be rewritten as $(x+2)^2+(y-1)^2=4$.

As C_1 and C_3 are concentric, centre of C_3 is $(-2,1)$.

Distance from $(-2,1)$ to AB $=\left|\frac{-2+1+3}{\sqrt{1^2+1^2}}\right|$
$=\frac{2}{\sqrt{2}}=\sqrt{2}$

∴ The equation of C_3 is $(x+2)^2+(y-1)^2=(\sqrt{2})^2=2$

20. (a) $x^2+y^2+4x+2y+4=0$
$$(x+2)^2+(y+1)^2=1$$

∴ The centre is $(-2,-1)$ and radius = 1.

(b) Distance from the centre to the line is
$$\left|\frac{a(-2)+b(-1)+c}{\sqrt{a^2+b^2}}\right|=\left|\frac{-2a-b+c}{\sqrt{a^2+b^2}}\right|.$$

Let the equation of the required tangent be $3x-4y+k=0$.
$$∴ \left|\frac{-6+4+k}{\sqrt{3^2+4^2}}\right|=1$$
$$\left|\frac{k-2}{5}\right|=1$$
$\frac{k-2}{5}=1$ or $\frac{k-2}{5}=-1$
$k=7$ or $k=-3$

∴ The equations of the required tangents are
$3x-4y+7=0$ and $3x-4y-3=0$.

(c) Slope of L is $\frac{3}{4}$.

The equation of the line passing through the centre of C and perpendicular to L is
$$\frac{y+1}{x+2}=\frac{-4}{3}$$
$$3y+3=-4x-8$$
$$4x+3y+11=0$$

The points of contact of the two tangents are given by solving

$\begin{cases}4x+3y+11=0 & .....(1)\\ 3x-4y+7=0 & .....(2)\end{cases}$ and $\begin{cases}4x+3y+11=0 & .....(3)\\ 3x-4y-3=0 & .....(4)\end{cases}$

$4\times(1)+3\times(2)$: $25x+65=0$, $x=\frac{-13}{5}$ $4\times(3)+3\times(4)$: $25x+35=0$, $x=\frac{-7}{5}$

Put into (1): $4\left(\frac{-13}{5}\right)+3y+11=0$, $y=\frac{-1}{5}$ Put into (3): $4\left(\frac{-7}{5}\right)+3y+11=0$, $y=\frac{-9}{5}$

∴ The points of contact are $\left(\frac{-13}{5}, \frac{-1}{5}\right)$ and $\left(\frac{-7}{5}, \frac{-9}{5}\right)$.

21. (a) Let the equation of the circle be
$$(x^2+y^2+4x-2y-11)+k(x+y-3)=0$$
$$x^2+y^2+(4+k)x+(k-2)y-(11+3k)=0$$

Centre $=\left(\frac{-4-k}{2}, \frac{-k+2}{2}\right)$

As AB is a diameter, $\left(\frac{-4-k}{2}\right)+\left(\frac{-k+2}{2}\right)=3$
$$-4-k-k+2=6$$
$$-2k=8$$
$$k=-4$$

∴ The equation of the required circle is $x^2+y^2-6y+1=0$.

(b) Let the equation of the required circle be
$$(x-4)^2+(y+4)^2=r$$
$$x^2+y^2-8x+8y+(32-r)=0.$$

The equation of the common chord is
$$(x^2+y^2+4x-2y-11)-(x^2+y^2-8x+8y+32-r)=0$$
$$12x-10y-43+r=0 \quad .....(*)$$

As this circle bisects the circumference of C, the common chord must be a diameter of C.

Hence, the centre of C lies on (*),
i.e. $12(-2)-10(1)-43+r=0$
$r=77$

∴ The equation of the required circle is $x^2+y^2-8x+8y-45=0$.

22. (a) Put $y=mx-2$ into the equation of C.
$$x^2+(mx-2)^2=1$$
$$(1+m^2)x^2-4mx+3=0 \quad .....(*)$$

∴ x_1, x_2 are roots of (*) and we have
$$x_1+x_2=\frac{4m}{1+m^2} \text{ and } x_1x_2=\frac{3}{1+m^2}$$

$$AB^2=(x_1-x_2)^2+(y_1-y_2)^2$$
$$=(x_1-x_2)^2+[(mx_1-2)-(mx_2-2)]^2$$
$$=(x_1-x_2)^2+m^2(x_1-x_2)^2=(1+m^2)(x_1-x_2)^2$$
$$=(1+m^2)[(x_1+x_2)^2-4x_1x_2]$$
$$=(1+m^2)\left[\frac{16m^2}{(1+m^2)^2}-\frac{12}{1+m^2}\right]$$
$$=\frac{4m^2-12}{1+m^2}=\frac{4(m^2-3)}{1+m^2}$$

$$∴ AB=2\sqrt{\frac{m^2-3}{1+m^2}}$$

(b) *(i)* If L meets C at two distinct points, then AB is real and AB > 0.

i.e. $m^2 - 3 > 0$

$(m - \sqrt{3})(m + \sqrt{3}) > 0$

$m > \sqrt{3}$ or $m < -\sqrt{3}$.

(ii) If L is a tangent to C, AB = 0

i.e. $m^2 - 3 = 0$

$m = \pm\sqrt{3}$.

(iii) If L does not meet C, AB is not real.

i.e. $m^2 - 3 < 0$

$(m - \sqrt{3})(m + \sqrt{3}) < 0$

$-\sqrt{3} < m < \sqrt{3}$.

(c) The two tangents are

$y = \sqrt{3}x - 2$ and $y = -\sqrt{3}x - 2$.

When $m = \sqrt{3}$, equation (*) becomes

$$4x^2 - 4\sqrt{3}x + 3 = 0$$
$$(2x - \sqrt{3})^2 = 0$$
$$x = \frac{\sqrt{3}}{2}$$

When $x = \frac{\sqrt{3}}{2}$, $y = \sqrt{3}\left(\frac{\sqrt{3}}{2}\right) - 2 = \frac{-1}{2}$

When $m = -\sqrt{3}$, equation (*) becomes

$$4x^2 + 4\sqrt{3}x + 3 = 0$$
$$(2x + \sqrt{3})^2 = 0$$
$$x = \frac{-\sqrt{3}}{2}$$

When $x = \frac{-\sqrt{3}}{2}$, $y = (-\sqrt{3})\left(\frac{-\sqrt{3}}{2}\right) - 2 = \frac{-1}{2}$

∴ The points of contact P, Q are $\left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$ and $\left(\frac{-\sqrt{3}}{2}, \frac{-1}{2}\right)$.

Equation of PQ is:

$$\frac{y + \frac{1}{2}}{x + \frac{\sqrt{3}}{2}} = \frac{\frac{-1}{2} - \left(\frac{-1}{2}\right)}{\frac{\sqrt{3}}{2} - \left(\frac{-\sqrt{3}}{2}\right)} = 0 \Rightarrow y + \frac{1}{2} = 0$$

∴ Equation of PQ is $y + \frac{1}{2} = 0$.

23. *(a)* Equation of the circle is:

$$(x - 0)(x - p) + (y - 1)(y - q) = 0$$
$$x^2 + y^2 - px - (1 + q)y + q = 0$$

When $y = 0$, we have

$$x^2 - px + q = 0 \qquad \ldots\ldots(*)$$

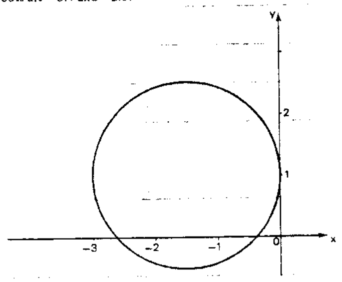
Hence, $\alpha + \beta = p$ and $\alpha\beta = q$.

(b) By comparing the equations $x^2 + 3x + 1 = 0$ and (*), we have

$p = -3$ and $q = 1$.

A circle with (0, 1) and (−3, 1) as ends of diameter is drawn as shown.

Hence, the roots are −0.4 and −2.6.



24. *(a)* *(i)* Equation of C_1 can be rewritten as

$$(x - 4)^2 + (y - 1)^2 = 1$$

Centre of C_1 is (4, 1) and radius = 1.

Distance from (4, 1) to x-axis = 1 = radius of C_1.

Hence, C_1 touches x-axis.

Further, the line joining (4, 1) and A is perpendicular to x-axis.

∴ C_1 touches x-axis at A.

(ii) Let the equation of OH be $y = mx$.

Put into the equation of C_1,

$$x^2 + (mx)^2 - 8x - 2(mx) + 16 = 0$$
$$(1 + m^2)x^2 - (8 + 2m)x + 16 = 0$$

By the condition of tangency,

$$[-(8 + 2m)]^2 - 4(16)(1 + m^2) = 0$$
$$64 + 32m + 4m^2 - 64 - 64m^2 = 0$$
$$32m - 60m^2 = 0$$
$$4m(8 - 15m) = 0$$
$$m = 0 \text{ or } m = \frac{8}{15}$$

∴ The equation of OH is $y = \frac{8}{15}x$.

(iii) By symmetry, slope of BH = $\frac{-8}{15}$.

Equation of BH is

$$\frac{y}{x - 8} = \frac{-8}{15}$$
$$15y = -8x + 64$$
$$8x + 15y = 64$$

(b) *(i)* Substitute $y = mx$ into the equation of C_2.

$$x^2 + (mx)^2 - 8x + 2f(mx) + c = 0$$
$$(1 + m^2)x^2 + (2fm - 8)x + c = 0$$

By the condition of tangency,

$$(2fm - 8)^2 - 4(1 + m^2)(c) = 0$$
$$4f^2m^2 - 32fm + 64 - 4c - 4cm^2 = 0$$
$$(4f^2 - 4c)m^2 - 32fm + (64 - 4c) = 0 \qquad \ldots\ldots(*)$$

As the x-axis and the line $4x + 3y = 0$ are tangents to C_2, the roots of (*) are 0 and $\frac{-4}{3}$.

Hence,
$$\begin{cases} 64 - 4c = 0 & \ldots\ldots(1) \\ \dfrac{32f}{4f^2 - 4c} = \dfrac{-4}{3} & \ldots\ldots(2) \end{cases}$$

From (1): $c = 16$.

Put into (2):

$$\frac{32f}{4f^2 - 64} = \frac{-4}{3}$$
$$96f = -16f^2 + 256$$
$$16f^2 + 96f - 256 = 0$$
$$16(f^2 + 6f - 16) = 0$$
$$16(f + 8)(f - 2) = 0$$
$$f = -8 \text{ (rejected) or } 2$$

(ii) Equation of AH is $x = 4$.

The coordinates of H are given by solving

$$\begin{cases} x = 4 \\ y = \dfrac{8}{15}x \end{cases}$$

i.e. H is $\left(4, \frac{32}{15}\right)$ and the area of ΔOBH is $\frac{1}{2} \times 8 \times \frac{32}{15} = \frac{128}{15}$

Similarly, AK ⊥ x-axis and the equation of AK is also $x = 4$.

The coordinates of K are given by

$$\begin{cases} x = 4 \\ 4x + 3y = 0 \end{cases}$$

i.e. K is $\left(4, \frac{-16}{3}\right)$ and the area of $\Delta OBK = \frac{1}{2} \times 8 \times \left(\frac{16}{3}\right) = \frac{64}{3}$

∴ area of ΔOBH : area of $\Delta OBK = \frac{128}{15} : \frac{64}{3} = 2 : 5$

25. *(a)* (C_1): $x^2 + y^2 - 24y = 0$

$$x^2 + (y - 12)^2 = 12^2$$

(C_2): $x^2 + y^2 + 24x - 6y + 144 = 0$

$$(x + 12)^2 + (y - 3)^2 = 3^2$$

∴ The centres of (C_1) and (C_2) are (0, 12) and (−12, 3) respectively, and the radii of (C_1) and (C_2) are respectively 12 and 3.

(b) Distance between the centres $= \sqrt{(0 + 12)^2 + (12 - 3)^2}$

$= 15$

= sum of radii of the two circles

∴ (C_1) and (C_2) touch each other.

(c) Distance from (0, 12) to x-axis = 12

= radius of (C_1)

Distance from (−12, 3) to x-axis = 3

= radius of (C_2)

Hence, the x-axis touches both (C_1) and (C_2) and it is a common tangent to the two circles.

(d) *(i)* P is the point of intersection of the x-axis and line of centres.

Equation of line of centre is:

$$\frac{y - 12}{x - 0} = \frac{12 - 3}{0 - (-12)} = \frac{9}{12} = \frac{3}{4}$$
$$y = \frac{3}{4}x + 12$$

When $y = 0$, $\frac{3}{4}x = -12$

$x = -16$

∴ P is (−16, 0).

(ii) As the other external tangent also passes through P, the equation of the external tangent can be written as

$$\frac{y - 0}{x + 16} = m$$
$$y = m(x + 16)$$

Put into the equation of (C_1):

$$x^2 + m^2(x + 16)^2 - 24m(x + 16) = 0$$
$$x^2(1 + m^2) + (32m^2 - 24m)x + (256m^2 - 384m) = 0$$

By the condition of tangency,

$$(32m^2 - 24m)^2 - 4(1 + m^2)(256m^2 - 384m) = 0$$
$$64m^2(4m - 3)^2 - 4(128m)(1 + m^2)(2m - 3) = 0$$
$$64m^2(16m^2 - 24m + 9) - 4(128m)(2m + 2m^3 - 3 - 3m^2) = 0$$
$$64m[16m^3 - 24m^2 + 9m - 16m - 16m^3 + 24 + 24m^2] = 0$$
$$64m[-7m + 24] = 0$$
$$m = 0 \text{ or } m = \frac{24}{7}.$$

∴ The equation of the other external tangent is $y = \frac{24}{7}(x + 16)$

$$= \frac{24}{7}x + \frac{384}{7}$$

(e) The third common tangent to the two circles is

$$(x^2 + y^2 - 24y) - (x^2 + y^2 + 24x - 6y + 144) = 0$$
$$-24x - 18y - 144 = 0$$
$$4x + 3y + 24 = 0$$