Ratio, Proportion and Variation

A. Rate

Rate compares two quantities of different kind where the value of one quantity corresponds to one unit of the other quantity.

e.g. walking speed (walk 12 km in 3 hrs) 4 km/h ($\text{speed} = \frac{\text{distance}}{\text{time}}$)

typing speed (type 80 words in 2 min) 40 words/min

"/" means 'per' or 'for each'

B. Ratio

Ratio relates two quantities of the same kind. The ratio of is expressed as $\frac{a}{b}$ or $a : b$ without unit.

e.g. ratio of boys to girls 21 : 20

ratio of 4 km to 2000 m 2 : 1

ratio of angles of a triangle 90° : 60° : 30° = 3 : 2 : 1 (continued ratio)

C. Proportion

A proportion is an equality of two ratios.

e.g. $a : b = 2 : 3$

$a : b = c :$ d or $\frac{a}{b} = \frac{c}{d}$ or $ad = bc$ ($b \neq 0$ and $d \neq 0$)

If 15 : 18 = 10 : x, find x. (12)

*Class Practice, p.105

Algebraic Manipulation of Ratios and Proportions

A. The k-method

This method is to express the variable in terms of a non-zero constant k.

e.g. $a : b = 2 : 3$

Take $a = 2k$ and $b = 3k$.

Examples:

1. If $a : b = 3 : 4$, find the ratio of $(2a - b) : (a + b)$. (2 : 11)
2. If $a : b : c = 3 : 5 : -1$ and $a - b + 2c = 10$, find a, b and c. $(-\frac{15}{2}, -\frac{25}{2}, \frac{5}{2})$
3. If $\frac{a}{b} = \frac{c}{d}$, prove that $1 - \frac{b^2}{a^2} = 1 - \frac{d^2}{c^2}$
4. If the interior angles of a quadrilateral are in the ratio 1 : 2 : 3 : 4, find the angles. (36°, 72°, 108°, 144°)
5. A tea blender mixes 3 kinds of tea costing respectively \$15, \$20 and \$30 per kg, in the ratio 2 : 3 : 1. What is the cost of 5 kg of the blended tea? (\$100)

B. Properties of Proportions

1) The Sum and Difference Property

If $a : b = c : d$, then $(a + b) : (a - b) = (c + d) : (c - d)$.

or If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

Example:

Find the values of x if $\frac{2x^2 - 4x - 3}{2x^2 + 4x - 3} = \frac{x+4}{x-4}$ (±1)

2) The Equal Ratios Property

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \cdots$, then each ratio $= \frac{pa + qc + re + \cdots}{pb + qd + rf + \cdots}$,

where p, q, r, … can be any numbers such that $pb + qd + rf + \ldots \neq 0$.

Examples:

1) If $\frac{a}{4} = \frac{b}{5} = \frac{ka + 3b}{27}$, find the value of k. (3)

2) If $\frac{a}{2} = \frac{b}{3} = \frac{c}{4} = \frac{5a - 6b + 7c}{x}$, find the value of x. (20)

C. Evaluation of Ratios under Given Conditions

Examples:

1) Given that $4x + 3y = 6x - 7y$. Find the ratio of $y : x$. (1 : 5)

2) Given that $2x - 3y + 6z = 0$ and $4x + y - 9z = 0$.

(a) Find $x : y : z$. (3 : 6 : 2)

(b) Find x, y and z if $2x^2 - y^2 + 5z^2 = 2$ (3, 6, 2 ; –3, –6, –2)

Direct Variation

When one quantity changes its value by a certain factor, the other quantity would change by the same factor. This relation is called direct variation.

e.g. For a car travelling at a uniform speed of 24 km/h

Time t (h)	1	2	3	4	5
Distance d (km)	24	48	72	96	120

This kind of relation is called directly variation.

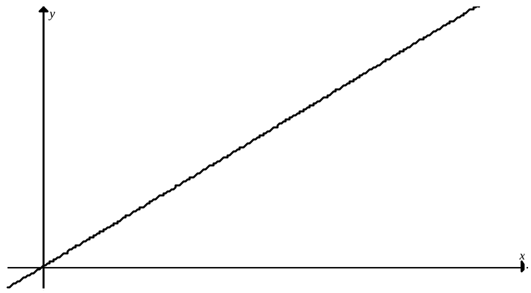
d varies directly as t : $d \propto t$

"$\propto$" means for ' varies directly as' .

$d = kt$ where k is a non-zero constant (variation constant).

In general, as variable y varies directly as variable x, we get $y = kx$.

The graph of y against x is a straight line passing through the origin.



Furthermore, in case that $y \propto x$, the relation between any two particular values of x_1 and x_2 with corresponding values of y_1 and y_2 is $\frac{y_1}{x_1} = \frac{y_2}{x_2}$.

Examples

1) If $y \propto (x^2 - 1)$, and $y = 32$ when $x = 5$, find

(a) the variation constant, $(k = \frac{4}{3})$

(b) the equation of variation, $(y = \frac{4}{3}(x^2 - 1))$

(c) the value of y when $x = 2$. (4)

2. Without finding the variation constant, find y when $x = 4$ if $y = 6$ when $x = 3$.
(8)

3. If $y \propto x^2$, find the percentage change in y if x is increased by 10%.
(21%)

Inverse Variation

When one quantity changes its value by a certain factor, the other quantity would decrease by the same factor. This relation is called inverse variation.

e.g. For a car travelling at a 100 km

Time t (h)	1/2	1	2	4	20
Speed v (km/h)	200	100	50	25	5

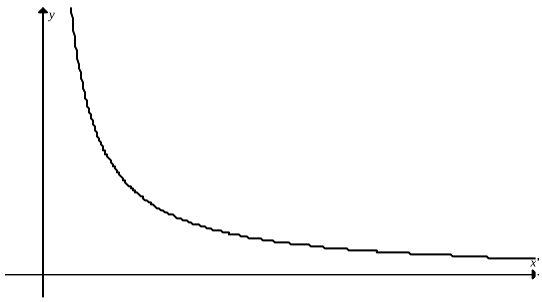
This kind of relation is called directly variation.

v varies directly as t : $v \propto \frac{1}{t}$

$vt = k$ where k is a non-zero constant (variation constant).

In general, as variable y varies inversely as variable x, we get $yx = k$.

The graph of y against x is a straight line passing through the origin.



Furthermore, in case that $y \propto \frac{1}{x}$, the relation between any two particular values of x_1 and x_2 with corresponding values of y_1 and y_2 is $x_1y_1 = x_2y_2$.

Example:

If y varies inversely as x and $y = 1$ when $x = 4$, find

(i) the variation constant. (4)

(ii) the value of x when y = 10. ($\frac{2}{5}$)

(iii) the percentage change in y when x is increase by 25%. (–25%)

Joint Variation

The relation such that a variable varies directly as the product of two or more other variables is called joint variation.

e.g. Area of a Triangle: $A \propto bh$

Volume of a Cone: $V \propto r^2h$

Example:

1) Express the statement "*H* varies as *x* and y^2 and inversely as *z*." into algebraic equation.

$(H = k\frac{xy^2}{y})$

*Class Practice, p.131

Example:

z varies as x^2 and inversely as *y*, and $z = 9$ when $x = 3$ and $y = 4$.

(a) Find the value of *x* when $z = 18$ and $y = 2$. $(z = 4\frac{x^2}{y}, x = \pm 3)$

(b) Find the percentage change in *z* if *x* is increased by 10% and *y* is decreased by 10%.

(34.44%)

Partial Variation

Partial Variation involves a sum in the equation of variation.

e.g. Perimeter of a Rectangle: $P = 2\ell + 2\omega$

Expenditure in a picnic: E = Rent of a Car + Cost of food

$E = k_1 + k_2 x$

Example:

1) Express the statement "*A* is partly constant and partly varies as *x*." into algebraic equation. $(A = k_1 + k_2 x)$

2) Express the statement "*Z* varies partly as *x* and partly inversely as y^2." into algebraic equation.

*Class Practice, p.131

Examples:

The total expense of a tea party is partly constant (spending on decoration) and partly varies as the number of participants (spending on food). The total expense is to be shared equally among all the participants. For 20 participants, the total expense is \$150, and for 35 participants, the total expense is \$225.

Find the amount each participant has to pay when there are 50 people coming to the party.

($E = 50 + 5x$, \$6)