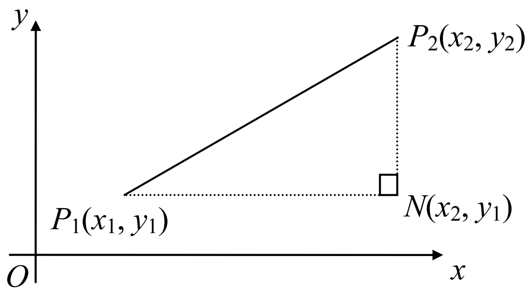
Coordinate Geometry

Distance Between Two Points

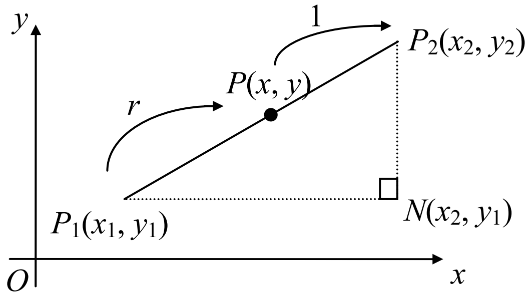
The distance between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is $P_1P_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.



Point of Division

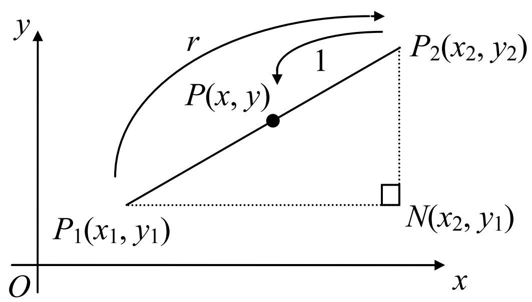
If $P(x, y)$ divides the line segment joining $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ internally in the ratio $\frac{P_1P}{PP_2} = r$, then the coordinate of P are given by

$x = \frac{x_1 + rx_2}{1 + r}$ and $y = \frac{y_1 + ry_2}{1 + r}$.

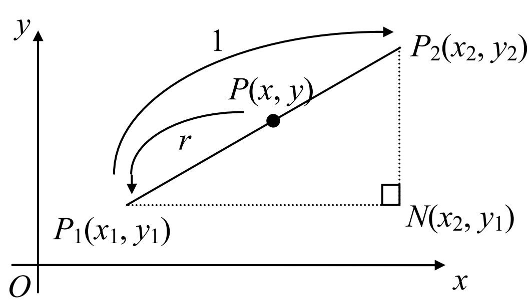


$r > 0$

If P lies on the produced line of P_1P_2, it is the external point of division and r is negative.



$r < -1$



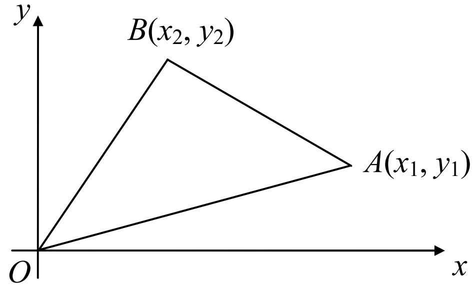
$-1 < r < 0$

If P is the midpoint of P_1P_2, then $r = 1$ and the coordinates of P are given by

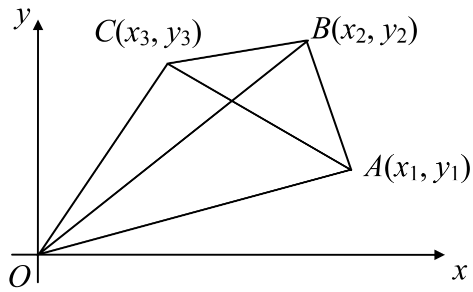
$x = \frac{x_1 + x_2}{2}$ and $y = \frac{y_1 + y_2}{2}$.

Area of Triangle

(a) Area of Triangle OAB = $\frac{1}{2}(x_1y_2 - x_2y_1)$



(b) Area of Triangle ABC = $\frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3) = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}$



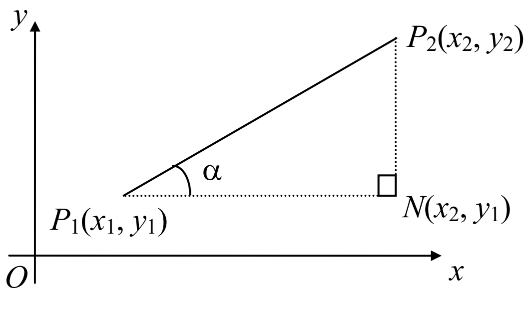
In general, the area obtained by the formula is positive if the vertices are taken in anti-clockwise direction, and the result is negative if the vertices are taken in clockwise direction.

(c) Area of Polygons = $\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{vmatrix}$

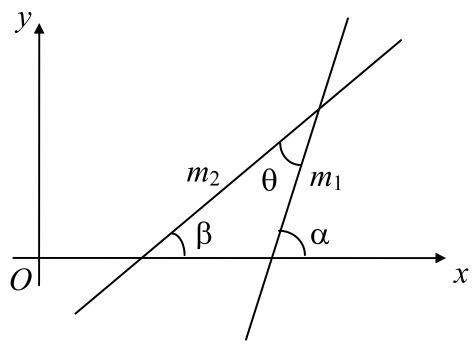
where $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ are taken in anti-clockwise direction.

Inclination and Slope

Slope $m = \tan\alpha = \frac{y_2 - y_1}{x_2 - x_1}$



Angles between Two Lines



$$\theta = \alpha - \beta$$
$$\tan\theta = \tan(\alpha - \beta)$$
$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$
$$= \frac{m_1 - m_2}{1 + m_1 m_2}$$

Since θ is an acute angle and $\tan\theta > 0$,

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

From above formula,
we get If $L_1 // L_2$ then $m_1 = m_2$
and If $L_1 \perp L_2$ then $m_1 m_2 = -1$.

Equations of Straight Lines

(A) Point-Slope Form

$$\frac{y - y_1}{x - x_1} = m$$

(B) Slope-Intercept Form

$$y = mx + c$$

(C) Two Point Form

$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

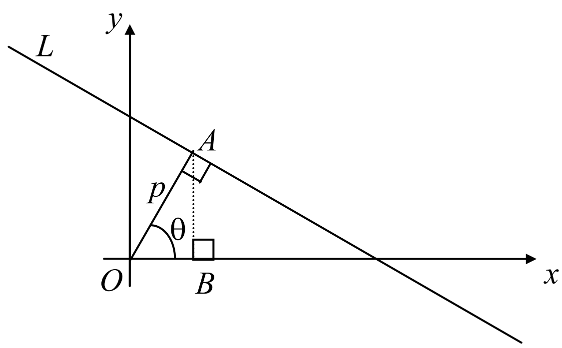
(D) Intercept Form

$$\frac{x}{a} + \frac{y}{b} = 1$$

(E) General Form

$$Ax + By + C = 0$$

Normal Form



OA = $p\cos\theta$
AB = $p\sin\theta$
where p is positive
Coordinates of $A = (p\cos\theta, p\sin\theta)$

Slope of OA = $\frac{p\sin\theta - 0}{p\cos\theta - 0} = \tan\theta$

Slope of the line L = $-\frac{1}{\tan\theta}$

Equation of the Line L:

$$\frac{y - p\sin\theta}{x - p\cos\theta} = -\frac{1}{\tan\theta}$$

$$y\tan\theta - p\sin\theta\tan\theta = -x + p\cos\theta$$

$$y\frac{\sin\theta}{\cos\theta} - p\sin\theta\frac{\sin\theta}{\cos\theta} = -x + p\cos\theta$$

$$y\sin\theta - p\sin^2\theta = -x\cos\theta + p\cos^2\theta$$

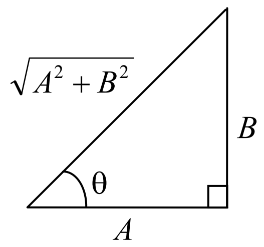
$$x\cos\theta + y\sin\theta - p = 0$$

where p (> 0) is the length of normal and θ ($0 < \theta < 2\pi$) is the inclination of normal.

Conversion of General Form to Normal Form

Comparing the General Form and Normal Form of the same straight line:

$$\begin{cases} Ax + By + C = 0 \\ x\cos\theta + y\sin\theta - p = 0 \end{cases}$$



Slope of the line = $-\frac{A}{B} = -\frac{1}{\tan\theta}$

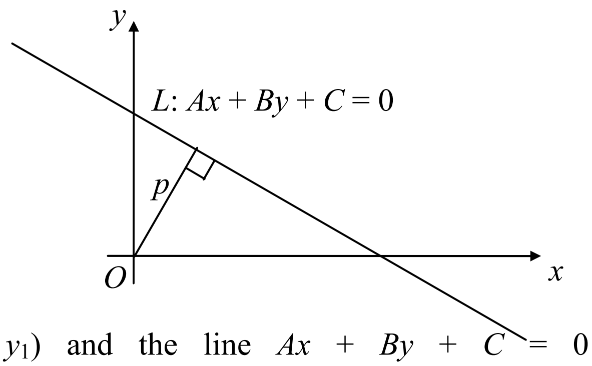
i.e. $\tan\theta = \frac{B}{A}$

$\sin\theta = \pm\frac{B}{\sqrt{A^2+B^2}}$ and $\cos\theta = \pm\frac{A}{\sqrt{A^2+B^2}}$

The equation in Normal Form is $\frac{Ax+By+C}{\pm\sqrt{A^2+B^2}}$.

Notice:

1) The sign we take must be opposite to that of C.
2) If C = 0, the sign we take must be same of B.
3) For the line $Ax + By + C = 0$, the perpendicular distance from the origin to the line is $p = \left|\frac{C}{\sqrt{A^2+B^2}}\right|$.



Distance Between a Point and a Line

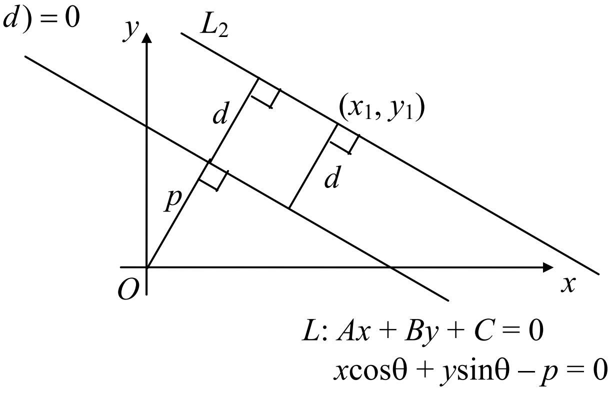
Let the distance between the point (x_1, y_1) and the line $Ax + By + C = 0$ ($x\cos\theta + y\sin\theta - p = 0$) be d.

The equation of the line L_2 which passes through (x_1, y_1) and parallel to $Ax + By + C = 0$ is $x\cos\theta + y\sin\theta - (p + d) = 0$.

Since (x_1, y_1) is on L_2, $x_1\cos\theta + y_1\sin\theta - (p + d) = 0$

$$d = x_1\cos\theta + y\sin\theta - p$$

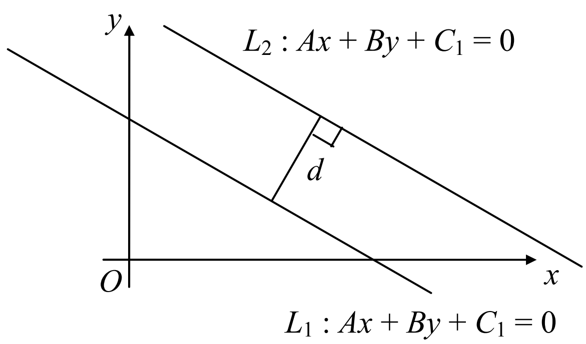
$$\therefore \quad = \left|\frac{Ax_1 + By_1 + C}{\sqrt{A^2+B^2}}\right|$$



Distance Between Two Parallel Lines

For two parallel lines $\begin{cases} Ax+By+C_1=0 \\ Ax+By+C_2=0 \end{cases}$,

there distance apart $d=\left|\dfrac{C_1-C_2}{\sqrt{A^2+B^2}}\right|$.



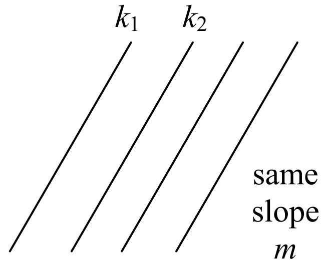
Family of Straight Lines

Family of straight lines are the lines whose have something in common.

I. Lines with Same Slope (m)

$y=mx+k$ or $Ax+By+k=0$ where k is a real number.

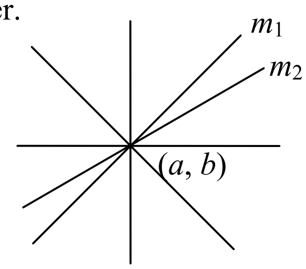
e.g. $\begin{cases} y=2x \\ y=2x+1 \\ y=2x-2 \\ \vdots \end{cases}$



2. Lines through Common Point (a, b)

$\dfrac{y-b}{x-a}=m$ or $y=m(x-a)+b$ where m is a real number.

e.g. $\begin{cases} y=x+1 \\ y=2x+1 \\ y=-3x+2 \\ \vdots \end{cases}$



3. Lines through Intersection Point of Two Straight Lines L_1 and L_2.

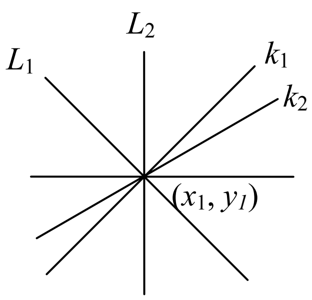
Given that $L_1 : A_1x+B_1y+C_1=0$

and $L_2 : A_2x+B_2y+C_2=0$.

The family of straight lines passing through the intersection point of L_1 and L_2 is

$L_1 : A_1x+B_1y+C_1+k(A_2x+B_2y+C_2)=0$ where k is a real number.

Each value of k will give an equation of one of the lines in the family.

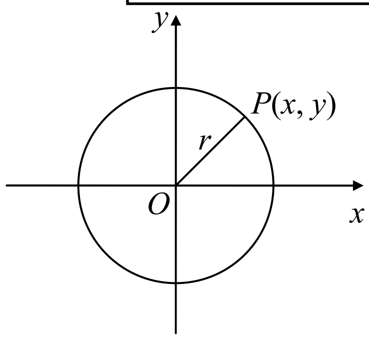


Circles

I. Circle with centre (0, 0) and radius r.
By distance formula

$$r^2 = (x-0)^2 + (y-0)^2$$

$$x^2 + y^2 = r^2$$

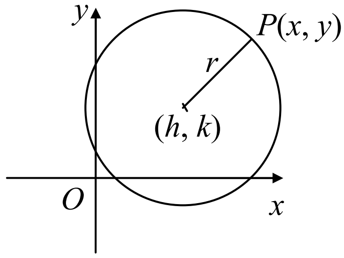


II. Circle with centre (h, k) and radius r.
By distance formula

$$r^2 = (x-h)^2 + (y-k)^2$$

$$= (x^2 - 2hx + h^2) + (y^2 - 2ky + k^2)$$

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$$



III. General Form

$$x^2 + y^2 + Dx + Ey + F = 0$$

Find Centre and Radius

Consider two equations $x^2 + y^2 + Dx + Ey + F = 0$ and $x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$.

$$\begin{cases} D = -2h \\ E = -2k \\ F = h^2 + k^2 - r^2 \end{cases}$$

$$\begin{cases} h = -\dfrac{D}{2} \\ E = -\dfrac{E}{2} \\ r = \sqrt{h^2 + k^2 - F} = \sqrt{\left(\dfrac{D}{2}\right)^2 + \left(\dfrac{E}{2}\right)^2 - F} = \dfrac{1}{2}\sqrt{D^2 + E^2 - 4F} \end{cases}$$

Equation of Circle with diameter ending at (x_1, y_1) and (x_2, y_2)

$$\left(\frac{y-y_1}{x-x_1}\right)\left(\frac{y-y_2}{x-x_2}\right) = -1$$

Intersection of a Line and a Circle

For a line $y = mx + c$ is a tangent to a circle $x^2 + y^2 + Dx + Ey + F = 0$.

If $\Delta > 0$, there are two points of intersection.
If $\Delta = 0$, there is only one point of intersection.
If $\Delta < 0$, there is no point of intersection.

Tangents to a Circle

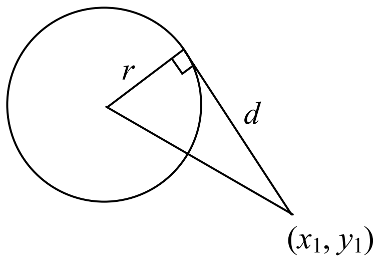
Conditions for a line $y = mx + c$ is a tangent to a circle $x^2 + y^2 + Dx + Ey + F = 0$.

1) $\Delta = 0$
2) Line from centre to the line = radius

I) Tangents with Given Slope
(C.D.)
II) Tangent at a Point on Its Circumference
(C.D.)
III) Tangents from External Point
(C.D.)

Length of Tangent

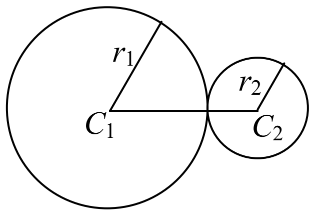
$$d = \sqrt{x_1^2 + y_1^2 + Dx_1 + Ey_1 + F}$$



Circles Touches Each Others

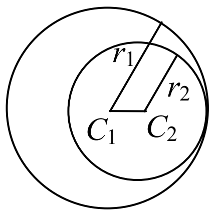
I. Touch Externally

$$C_1C_2 = r_1 + r_2$$



II. Touch Internally

$$C_1C_2 = r_1 - r_2$$



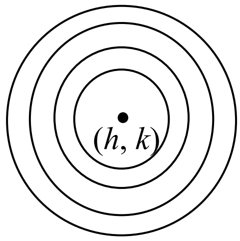
Family of Circles

I. Concentric Circles

$x^2 + y^2 - 2hx - 2ky - F = 0$ where F is a real number.

or $(x-h)^2 + (y-k)^2 = r^2$ where r is a real number.

e.g. $x^2 + y^2 - 2x + y - F = 0$

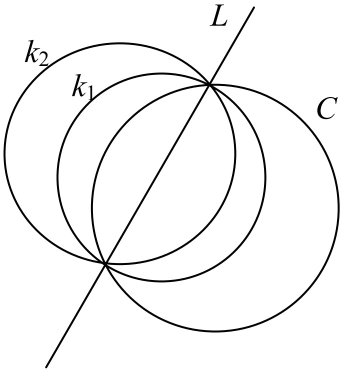


II. Circles Through Intersection of a Line $L: Ax + By + C = 0$ and a Circle $C: x^2 + y^2 + Dx + Ey + F = 0$.

$x^2 + y^2 + Dx + Ey + F + k(Ax + By + C) = 0$

where k is a real number.

Or $x^2 + y^2 + (D + kA)x + (E + kB)y + (F + kC) = 0$



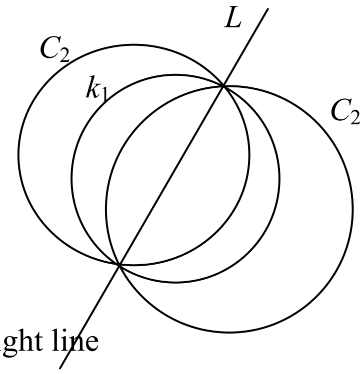
II. Circles Through Intersection of a Circle 1 $C_1: x^2 + y^2 + D_1x + E_1y + F_1 = 0$ and a Circle 2 $C_2: x^2 + y^2 + D_2x + E_2y + F_2 = 0$.

$x^2 + y^2 + D_1x + E_1y + F_1 + k(x^2 + y^2 + D_2x + E_2y + F_2) = 0$

where k is a real number.

Or

$(1+k)x^2 + (1+k)y^2 + (D_1 + kD_2)x + (E_1 + kE_2)y + (F_1 + kF_2) = 0$



When $k = -1$, the equation above becomes an equation of a straight line

$L: (D_1 - D_2)x + (E_1 - E_2)y + (F_1 - F_2) = 0$

If there are two points of intersection, L is the common chord of the two given circles.

If there is only one point of intersection, L is the common tangent of the two given circles.

If there is no point of intersection, L is the radial axis of the two given circles.

The family of circle through the intersection of a C_1 and C_2 can be reduced to the family of circle through the intersection of C_1 or C_2 and its common chord L.

i.e. $x^2 + y^2 + D_1x + E_1y + F_1 + k[(D_1 - D_2)x + (E_1 - E_2)y + (F_1 - F_2)] = 0$

or $x^2 + y^2 + D_2x + E_2y + F + k[(D_1 - D_2)x + (E_1 - E_2)y + (F_1 - F_2)] = 0$

Locus Problem

Locus is the path described when a point moves on a plane under certain condition. If the coordinates of the moving point is (x, y), the relationship between x and y, i.e. $F(x, y) = 0$ is the equation of the locus.