Review Notes for EC750

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1st Dec 07

Contents

Preface

This is note for Macro.

Introduction

Notes for Macro. It would include EC750 and EC751 which is core for comprehensive examination of Economics Ph.D in Boston College.

Part I

EC720 Math

Chapter 1

Two Useful Theorems

1.1 The Kuhn-Tucker Theorem

- Maximization
  - Problem Setup: (k constraints and n choice variable)

$$\begin{array}{lll} \max\limits_{x_1,\ldots,x_n} & F(x_1,\ldots,x_n) & \\ s.t. & c_i \geq G_i(x_1,\ldots,x_n) & \text{for all } i=1,\ldots,k \end{array}$$

  - Lagrangian:

$$\mathcal{L}(x_1,\ldots,x_n;\lambda_1,\ldots,\lambda_k) = F(x_1,\ldots,x_n) + \sum_{i=1}^{k} \lambda_i \left[c_i - G_i(x_1,\ldots,x_n)\right]$$

  - Kuhn-Tucker conitions

$$\begin{array}{rl} \dfrac{\partial\mathcal{L}}{\partial x_i} = F_i + \sum_{m=1}^{k} \lambda_m \dfrac{\partial G_m}{\partial x_i} = 0 & \text{for } i=1,\ldots,n \\ \dfrac{\partial\mathcal{L}}{\partial\lambda} = c_i - G_i(x_1,\ldots,x_n) \geq 0 & \text{for } i=1,\ldots,n \\ \lambda_i \geq 0; \quad \lambda_i\left[c_i - G_i(x_1,\ldots,x_n)\right] = 0 & \text{for } i=1,\ldots,n \end{array}$$

    - Necessary conditions only
    - Constraint qualification: Hessian $|\partial G_i(x^*)/\partial x_j|$ has maximum rank or simply $G'(x^*) \neq 0$ in one constraint case:

- Minimization
  - Problem Setup: (k constraints and n choice variable)

$$\begin{array}{lll} \min\limits_{x_1,\ldots,x_n} & F(x_1,\ldots,x_n) & \\ s.t. & G_i(x_1,\ldots,x_n) \geq c_i & \text{for all } i=1,\ldots,k \end{array}$$

– Lagrangian:

$$\mathcal{L}(x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_k) = F(x_1, \ldots, x_n) - \sum_{i=1}^{k} \lambda_i [G_i(x_1, \ldots, x_n) - c_i]$$

– Kuhn-Tucker conitions (Proof. for single variable case is in appendix)

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_i} = F_i - \sum_{m=1}^{k} \lambda_m \frac{\partial G_m}{\partial x_i} &= 0 && \text{for } i = 1, \ldots, n \\ \frac{\partial \mathcal{L}}{\partial \lambda} = -G_i(x_1, \ldots, x_n) + c_i &\leq 0 && \text{for } i = 1, \ldots, n \\ \lambda_i \geq 0; \quad \lambda_i [c_i - G_i(x_1, \ldots, x_n)] &= 0 && \text{for } i = 1, \ldots, n \end{aligned}$$

1.2 The Envelope Theorem

- Objective functions with parameters:

$$F(x_1, \ldots, x_n; \theta_1, \ldots, \theta_h)$$

- Value functions

$$V(\theta_1, \ldots, \theta_n) = \max_{x_1, \ldots, x_n} F(x_1, \ldots, x_n; \theta_1, \ldots, \theta_h)$$

or

$$V(\theta_1, \ldots, \theta_n) = \min_{x_1, \ldots, x_n} F(x_1, \ldots, x_n; \theta_1, \ldots, \theta_h)$$

- Envelope Theorem (with constraint)

  – assumption on the *existence* of an *unique* solution $x^*(\theta_1, \ldots, \theta_n)$ to optimization problem

  – constraint qualification: $dG_i[x^*(\theta), \theta]/dx \neq 0$ for all value of θ and for all i.

  – envelope theorem implies

$$\frac{dV(\theta_1, \ldots, \theta_n)}{d\theta_i} = \frac{\partial \mathcal{L}}{\partial \theta_i} \quad \text{for } i = 1, \ldots, n$$

  – remark: For maximization problem

$$\frac{dV(\theta_1, \ldots, \theta_n)}{d\lambda_i} = c_i - G_i \quad \text{for } i = 1, \ldots, n$$

  and for minimization problem

$$\frac{dV(\theta_1, \ldots, \theta_n)}{d\lambda_i} = -c_i + G_i \quad \text{for } i = 1, \ldots, n$$

Chapter 2

Dynamic Optimization

2.1 Discrete Case

- Setup
  - T period of time but there is $T+1$ point of time, denoted as $t = 0, 1, \ldots, T$
  - Stock variable: y_t (defined as value beginning of the period , from $t = 0$ to $t = T+1$)
  - flow variable: z_t (defined as value during period, from $t = 0$ to $t = T$
  - Objective function: (additively separable with discount factor)

    $$\sum_{t=0}^{T} \beta^t F\left(y_t, z_t; t\right)$$

    where $0 < \beta \leq 1$.
  - Evolution of stock variables (only $T+1$ flow variables so that only $T+1$ envolution)

    $$Q\left(y_t, z_t; t\right) \geq y_{t+1} - y_t \qquad \text{for all } t = 0, 1, \ldots, T$$

  - Constraints of each period (same as the number of envolution as constraint is to restrict the process)

    $$c \geq G\left(y_t, z_t\right) \qquad \text{for all } t = 0, 1, \ldots, T$$

  - Initial and terminal condition

    $$y_0 \text{ is given}$$
    $$y_{T+1} \geq y^*$$

– Formal problem:

$$\begin{array}{ll}\max_{\{z_t\}_{t=0}^{T},\{y_t\}_{t=1}^{T+1}} & \sum_{t=0}^{T}\beta^t F(y_t,z_t;t) \\ s.t. & y_t + Q(y_t,z_t;t) \geq y_{t+1} \quad \text{for all } t=0,1,\ldots,T \\ & c \geq G(y_t,z_t;t) \quad \text{for all } t=0,1,\ldots,T \\ & y_0 \text{ is given} \\ & y_{T+1} \geq y^* \end{array}$$

- The Kuhn-Tucker Formulation

  – The Lagrangrian is

$$\begin{aligned}\mathcal{L} = & \sum_{t=0}^{T}\beta^t F(y_t,z_t;t) + \sum_{t=0}^{T}\pi_{t+1}\left[y_t + Q(y_t,z_t;t) - y_{t+1}\right] \\ & + \sum_{t=0}^{T}\lambda_t\left[c - G(y_t,z_t;t)\right] + \phi\left[y_{T+1} - y^*\right]\end{aligned}$$

  – FOCs are

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial y_t} &= \beta^t F_{y_t} + \pi_{t+1}(1+Q_{y_t}) - \pi_t - \lambda_t G_{y_t} = 0 \quad \text{for all } t=1,2,\ldots,T \\ \frac{\partial\mathcal{L}}{\partial z_t} &= \beta^t F_{z_t} + \pi_{t+1}Q_{z_t} - \lambda_t G_{z_t} = 0 \quad \text{for all } t=0,1,\ldots,T \\ \frac{\partial\mathcal{L}}{\partial y_{T+1}} &= \pi_{T+1} - \phi = 0 \\ \frac{\partial\mathcal{L}}{\partial \pi_{t+1}} &= y_t + Q(y_t,z_t;t) - y_{t+1} \geq 0 \quad \text{for all } t=0,1,\ldots,T \\ \frac{\partial\mathcal{L}}{\partial \lambda_t} &= c - G(y_t,z_t;t) \geq 0 \quad \text{for all } t=0,1,\ldots,T \\ \pi_{t+1} &\geq 0 \quad \pi_{t+1}\left[y_t + Q(y_t,z_t;t) - y_{t+1}\right] = 0 \quad \text{for all } t=0,1,\ldots,T \\ \lambda_t &\geq 0 \quad \lambda_t\left[c - G(y_t,z_t;t)\right] = 0 \quad \text{for all } t=0,1,\ldots,T \\ \phi &\geq 0 \quad \phi(y_{T+1} - y^*) = 0\end{aligned}$$

  – Assuming constraint for envolution is binding, so that we have

$$y_t + Q(y_t,z_t;t) - y_{t+1} = 0 \quad \text{for all } t=0,1,\ldots,T$$

  and we also have

$$\pi_{T+1}(y_{T+1} - y^*) = 0$$

  – Four equations with four unknowns y_t, z_t, π_t and λ_t:

$$\left\{\begin{array}{c}\beta^t F_{y_t} + \pi_{t+1}(1+Q_{y_t}) - \pi_t - \lambda_t G_{y_t} = 0 \quad \text{for all } t=1,2,\ldots,T \\ \beta^t F_{z_t} + \pi_{t+1}Q_{z_t} - \lambda_t G_{z_t} = 0 \quad \text{for all } t=0,1,\ldots,T \\ y_t + Q(y_t,z_t;t) - y_{t+1} = 0 \quad \text{for all } t=0,1,\ldots,T \\ \lambda_t\left[c - G(y_t,z_t;t)\right] = 0 \quad \text{for all } t=0,1,\ldots,T\end{array}\right.$$

with initial and terminal conditions

$$\begin{aligned} &y_0 \text{ is given} \\ &\pi_{T+1}\left(y_{T+1} - y^*\right) = 0 \end{aligned}$$

– Remark: when $T \to \infty$, the transversality condition would then be

$$\lim_{T\to\infty} \pi_{T+1}\left(y_{T+1} - y^*\right) = 0$$

- Maximum Principle
  - it refers to the fact that solving the Hamiltonian is same as solving by Kuhn-Tuker method
  - Hamiltonian

$$\begin{aligned} H\left(y_t, \pi_{t+1}\right) &= \max_{z_t} \beta^t F\left(y_t, z_t; t\right) + \pi_{t+1} Q\left(y_t, z_t; t\right) \\ s.t. \quad c &\geq G\left(y_t, z_t; t\right) \end{aligned}$$

  - The FOC of maximization problem is

$$\frac{\partial H}{\partial z_t} = \beta^t F_{z_t} + \pi_{t+1} Q_{z_t} - \lambda_t G_{z_t} = 0 \quad \text{for all } t = 0, 1, \ldots, T$$

  - By Envelope theorem, we have

$$\begin{aligned} \pi_{t+1} - \pi_t &= -\frac{\partial H}{\partial y_t} = -\beta^t F_{y_t} - \pi_{t+1} Q_{y_t} + \lambda_t G_{y_t} \quad \text{for all } t = 1, 2, \ldots, T \\ y_{t+1} - y_t &= \frac{\partial H}{\partial \pi_{t+1}} = Q\left(y_t, z_t; t\right) \quad \text{for all } t = 0, 1, \ldots, T \end{aligned}$$

2.2 Continuous Case

Setup

1. time $t \in [0, T]$, where T can be finite or infinite
2. Stock variable: $y\left(t\right)$
3. flow variable: $z\left(t\right)$
4. Objective function: (additively separable with discount factor)

$$\int_{t=0}^{T} e^{-\rho t} F\left(y_t, z_t, t\right) dt$$

where $\rho \geq 0$.

5. Evolution of stock variables:

$$Q\left(y\left(t\right),z\left(t\right),t\right)\Delta t \geq y\left(t+\Delta t\right)-y\left(t\right)$$

   or when $\Delta t \to 0$,

$$Q\left(y\left(t\right),z\left(t\right),t\right) \geq \dot{y}$$

6. Constraints at each point of time,

$$c \geq G\left(y\left(t\right),z\left(t\right),t\right)$$

7. Initial and terminal condition

$$\begin{aligned} & y\left(0\right) \text{ is given} \\ & y\left(T\right) \geq y^{*} \end{aligned}$$

8. Formal problem:

$$\begin{array}{cl} \max\limits_{\{z(t)\}_{t=0}^{T},\{y(t)\}_{t=0}^{T}} & \displaystyle\int_{t=0}^{T} e^{-\rho t} F\left(y\left(t\right),z\left(t\right),t\right)dt \\ s.t. & Q\left(y\left(t\right),z\left(t\right),t\right) \geq y\left(t\right) \quad \text{for all } t \in [0,T] \\ & c \geq G\left(y_t, z_t, t\right) \quad \text{for all } t \in [0,T] \\ & y\left(0\right) \text{ is given} \\ & y\left(T\right) \geq y^{*} \end{array}$$

The Kuhn-Tucker Formulation

The Lagrangrian is

$$\begin{aligned} \mathcal{L} = & \int_{t=0}^{T} e^{-\rho t} F\left(y\left(t\right),z\left(t\right),t\right)dt + \int_{t=0}^{T} \pi\left(t\right)\left[Q\left(y\left(t\right),z\left(t\right),t\right)-\dot{y}\left(t\right)\right]dt \\ & + \int_{t=0}^{T} \lambda\left(t\right)\left[c-G\left(y\left(t\right),z\left(t\right),t\right)\right] + \phi\left[y\left(T\right)-y^{*}\right]dt \end{aligned}$$

From Integration by parts,

$$\begin{aligned} \int_{0}^{T}\left\{\frac{d}{dt}\left[\pi\left(t\right)y\left(t\right)\right]\right\}dt &= \int_{0}^{T}\dot{\pi}\left(t\right)y\left(t\right)dt + \int_{0}^{T}\pi\left(t\right)\dot{y}\left(t\right)dt \\ \Rightarrow \quad \pi\left(T\right)y\left(T\right)-\pi\left(0\right)y\left(0\right) &= \int_{0}^{T}\dot{\pi}\left(t\right)y\left(t\right)dt + \int_{0}^{T}\pi\left(t\right)\dot{y}\left(t\right)dt \\ \Rightarrow \quad -\int_{0}^{T}\dot{\pi}\left(t\right)y\left(t\right)dt &= \int_{0}^{T}\pi\left(t\right)\dot{y}\left(t\right)dt + \pi\left(0\right)y\left(0\right)-\pi\left(T\right)y\left(T\right) \end{aligned}$$

so that

$$\begin{aligned}\mathcal{L} &= \int_{t=0}^{T} e^{-\rho t} F\left(y\left(t\right), z\left(t\right), t\right) dt + \int_{t=0}^{T} \pi\left(t\right) \left[Q\left(y\left(t\right), z\left(t\right), t\right)\right] dt \\ &\quad + \int_{0}^{T} \pi\left(t\right) \dot{y}\left(t\right) dt + \pi\left(0\right) y\left(0\right) - \pi\left(T\right) y\left(T\right) \\ &\quad + \int_{t=0}^{T} \lambda\left(t\right) \left[c - G\left(y\left(t\right), z\left(t\right), t\right)\right] dt + \phi\left[y\left(T\right) - y^{*}\right]\end{aligned}$$

FOCs are

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial z\left(t\right)} &= e^{-\rho t} F_z + \pi\left(t\right) Q_z - \lambda\left(t\right) G_z = 0 \quad \text{for all } t \in [0, T] \\ \frac{\partial \mathcal{L}}{\partial y\left(t\right)} &= e^{-\rho t} F_y + \pi\left(t\right) Q_y + \dot{\pi}\left(t\right) - \lambda\left(t\right) G_y = 0 \quad \text{for all } t \in (0, T) \\ \frac{\partial \mathcal{L}}{\partial y\left(t\right)} &= e^{-\rho T} F_y + \pi\left(T\right) Q_y + \dot{\pi}\left(T\right) - \pi\left(T\right) - \lambda\left(t\right) G_y + \phi = 0 \quad \text{for all } t = T \\ \frac{\partial \mathcal{L}}{\partial \pi\left(t\right)} &= Q\left(y\left(t\right), z\left(t\right), t\right) - \dot{y}\left(t\right) \geq 0 \quad \text{for all } t \in (0, T) \\ \frac{\partial \mathcal{L}}{\partial \lambda\left(t\right)} &= c - G\left(y\left(t\right), z\left(t\right), t\right) \geq 0 \quad \text{for all } t = [0, T] \\ \pi\left(t\right) &\geq 0 \quad \pi\left(t\right)\left[Q\left(y\left(t\right), z\left(t\right), t\right) - \dot{y}\left(t\right)\right] = 0 \quad \text{for all } t = (0, T) \\ \lambda\left(t\right) &\geq 0 \quad \lambda\left(t\right)\left[c - G\left(y\left(t\right), z\left(t\right), t\right)\right] = 0 \quad \text{for all } t = [0, T] \\ \phi &\geq 0 \quad \phi\left(y\left(T\right) - y^{*}\right) = 0\end{aligned}$$

Further assume all functions of t are continuously differentiable, then

$$e^{-\rho t} F_y + \pi\left(t\right) Q_y + \dot{\pi}\left(t\right) - \lambda\left(t\right) G_y = 0 \quad \text{for all } t \in [0, T]$$

or

$$\dot{\pi}\left(t\right) = -\left[e^{-\rho t} F_y + \pi\left(t\right) Q_y - \lambda\left(t\right) G_y\right]$$

so that

$$\pi\left(T\right) = \phi.$$

Assuming constraint for envolution is binding, so that we have

$$\dot{y}\left(t\right) = Q\left(y\left(t\right), z\left(t\right), t\right) \quad \text{for all } t = [0, T]$$

Four equations with four unknowns $y\left(t\right), z\left(t\right), \pi\left(t\right)$ and $\lambda\left(t\right)$:

$$\begin{cases} e^{-\rho t} F_z + \pi\left(t\right) Q_z - \lambda\left(t\right) G_z = 0 \quad \text{for all } t \in [0, T] \\ \dot{\pi}\left(t\right) = -\left[e^{-\rho t} F_y + \pi\left(t\right) Q_y - \lambda\left(t\right) G_y\right] \quad \text{for all } t = [0, T] \\ \dot{y}\left(t\right) = Q\left(y\left(t\right), z\left(t\right), t\right) \quad \text{for all } t = [0, T] \\ \lambda\left(t\right)\left[c - G\left(y\left(t\right), z\left(t\right), t\right)\right] = 0 \quad \text{for all } t = [0, T] \end{cases}$$

with initial and terminal conditions

$$\begin{aligned} & y(0) \text{ is given} \\ & \pi(T)\left[y(T)-y^*\right]=0 \end{aligned}$$

Remark: when $T \to \infty$, the transversality condition would then be

$$\lim_{T\to\infty} \pi(T)\left[y(T)-y^*\right]=0$$

Maximum Principle

It refers to the fact that solving the Hamiltonian is same as solving by Kuhn-Tuker method
The Hamiltonian is

$$\begin{aligned} H(y(t),\pi(t)) &= \max_{z(t)} e^{-\rho t}F(y(t),z(t),t)+\pi(t)Q(y(t),z(t),t) \\ s.t. \quad c &\geq G(y(t),z(t);t) \end{aligned}$$

The FOC of maximization problem is

$$\frac{\partial H}{\partial z(t)}=e^{-\rho t}F_z+\pi(t)Q_z-\lambda(t)G_z=0 \quad \text{for all } t=[0,T]$$

By Envelope theorem, we have

$$\begin{aligned} \frac{dH}{dy(t)} &= e^{-\rho t}F_y+\pi(t)Q_y-\lambda G_y \\ \frac{dH}{d\pi(t)} &= Q(y(t),z(t),t) \end{aligned}$$

and now compared this with result from Kuhn-Tucker, we have

$$\begin{aligned} \dot{\pi}(t) &= -\frac{dH}{dy(t)}=-\left[e^{-\rho t}F_y+\pi(t)Q_y-\lambda(t)G_y\right] \quad \text{for all } t=[0,T] \\ \dot{y}(t) &= \frac{dH}{d\pi(t)}=Q(y(t),z(t),t) \quad \text{for all } t=[0,T] \end{aligned}$$

2.3 Final Note

1. Present-Value Hamiltonian versus Current Value Hamiltonian
2. Phase Diagram

Chapter 3

Dynamic Optimization

3.1 Perfect Foresight in discrete Case

Setup

1. no uncertainty
2. T period of time but there is $T+1$ point of time, denoted as $t = 0, 1, \ldots$
3. Stock variable: y_t (defined as value beginning of the period , from $t = 0$)
4. flow variable: z_t (defined as value during period, from $t = 0$)
5. Objective function: (additively separable with discount factor)

   $$\sum_{t=0}^{T} \beta^t F(y_t, z_t; t)$$

   where $0 < \beta \leq 1$.
6. Evolution of stock variables (only $T+1$ flow variables so that only $T+1$ envolution)

   $$Q(y_t, z_t; t) \geq y_{t+1} - y_t \qquad \text{for all } t = 0, 1, \ldots, T$$

7. Constraints of each period (same as the number of envolution as constraint is to restrict the process)

   $$c \geq G(y_t, z_t) \qquad \text{for all } t = 0, 1, \ldots, T$$

8. Initial condition

   $$y_0 \text{ is given}$$

9. Formal problem:

$$\begin{aligned} \max_{\{z_t\}_{t=0}^{T}, \{y_t\}_{t=1}^{T+1}} \quad & \textstyle\sum_{t=0}^{T} \beta^t F(y_t, z_t; t) \\ s.t. \quad & y_t + Q(y_t, z_t; t) \geq y_{t+1} \quad \text{for all } t = 0, 1, \ldots, T \\ & c \geq G(y_t, z_t; t) \quad \text{for all } t = 0, 1, \ldots, T \\ & y_0 \text{ is given} \end{aligned}$$

Kuhn-Tucker Formulation

Bellman Equation

3.2 Dynamic Stochastic in discrete Case

Setup

Bellman Equation

Appendix A

Appendix I EC740

A.1 Proof of Kuhn Tucker Condition for minimization problem with single choice variable and single constraint:

Suppose x^* minimize $F(x)$ subject to $G(x) \geq c$. Given $G'(x^*) \neq 0$ and the lagrangian to be

$$\mathcal{L} = F(x) - \lambda [G(x) - c],$$

then there exists λ and x^* such that

$$\mathcal{L}_x = F'(x^*) - \lambda G'(x^*) = 0 \tag{A.1}$$

$$\mathcal{L}_\lambda = -G(x^*) + c \leq 0 \tag{A.2}$$

$$\lambda \geq 0 \tag{A.3}$$

$$\lambda [G(x^*) - c] = 0 \tag{A.4}$$

Case 1: Non-Binding constraint
Non-binding constraint means $G(x^*) > c$ so that (2) is true. From (4), we know that $\lambda = 0$. Then (3) is also true. Finally, we have to show (1), which is $F'(x^*) = 0$. Suppose $F'(x^*) < 0$, there exists $\varepsilon > 0$ such that $F(x^* + \varepsilon) < F(x^*)$ but $G(x^* + \varepsilon) > c$, which violates the fact that x^* is minimizer. Similarly, suppose $F'(x^*) > 0$, there exists $\varepsilon > 0$ such that $F(x^* - \varepsilon) < F(x^*)$ but $G(x^* - \varepsilon) < c$, which viotlates the optimality of x^*.
Case 2: Binding constraint
Binding constraint means $G(x^*) = c$ so that (2) and (4) are true. Then from (1), given $G'(x) \neq 0$, we have $\lambda = F'(x^*)/G'(x^*)$ so that by (3), we have

$$\lambda = \frac{F'(x^*)}{G'(x^*)} \geq 0.$$

Suppose the contrary that $\lambda < 0$, so we have two cases: either

$$\begin{cases} F'(x^*) > 0 \\ G'(x^*) < 0 \end{cases} \text{ or } \begin{cases} F'(x^*) < 0 \\ G'(x^*) > 0 \end{cases}$$

In the first case, there would exist ε such that $F(x^* - \varepsilon) < F(x^*)$ and $G(x^* - \varepsilon) > G(x^*) = c$, which means $x^* - \varepsilon$ is better minimizer. Contradiction. In the second case, there would also exist ε such that $F(x^* + \varepsilon) < F(x^*)$ and $G(x^* + \varepsilon) > G(x^*) = c$, which implies means $x^* + \varepsilon$ is better minimizer. Contradiction. Hence (1) is also true.

A.2 Summary of EC750

Kuhn-Tuker and Envelop Theorem

Static Maximization Problem	
$\max_x F(x) \quad \text{s.t. } c \geq G(x)$	
Kuhn-Tucker Theorem: Assumption $G'(x^*) \neq 0$	
Lagrangian	$L = F(x) + \lambda[c - G(x)]$
First order condition	$L_x = 0 \Rightarrow F_x - \lambda G_x = 0$
constraint	$L_\lambda \geq 0 \Rightarrow c - G(x) \geq 0$
Non-negativity condition	$\lambda \geq 0$
Complementary slackness	$\lambda[c - G(x)] = 0$
Envelope Theorem: Assumption $G_1[x^*(\theta), \theta] \neq 0$	
Maximum value function	$V(\theta) = \max_x F(x, \theta) \quad \text{s.t. } c \geq G(x, \theta)$
optimal value	$x^*(\theta) = \arg\max_x F(x, \theta) \quad \text{s.t. } c \geq G(x, \theta)$
associated multiplier	$\lambda^*(\theta)$
Envelope Theorem	$V'(\theta) = F_2[x^*(\theta), \theta] - \lambda^*(\theta) G_2[x^*(\theta), \theta]$

Static Minimization Problem	
$\min_x F(x) \quad \text{s.t. } G(x) \geq c$	
Kuhn-Tucker Theorem: Assumption $G'(x^*) \neq 0$	
Lagrangian	$L = F(x) - \lambda[G(x) - c]$
First order condition	$L_x = 0 \Rightarrow F_x - \lambda G_x = 0$
constraint	$L_\lambda \leq 0 \Rightarrow -G(x) - c \leq 0$
Non-negativity condition	$\lambda \geq 0$
Complementary slackness	$\lambda[c - G(x)] = 0$
Envelope Theorem: Assumption $G_1[x^*(\theta), \theta] \neq 0$	
Mnimum value function	$V(\theta) = \min F(x, \theta) \quad \text{s.t. } G(x, \theta) \geq c$
optimal value	$x^*(\theta) = \arg\min_x F(x, \theta) \quad \text{s.t. } G(x, \theta) \geq c$
associated multiplier	$\lambda^*(\theta)$
Envelope Theorem	$V'(\theta) = F_2[x^*(\theta), \theta] + \lambda^*(\theta) G_2[x^*(\theta), \theta]$

Maximium Principle

Discrete Dynamic Maximization problem	
$\max_{\{z(t)\}_{t=0}^{T},\{y(t)\}_{t=1}^{T}} \quad \sum_{t=0}^{T} \beta^t F(y(t), z(t), t)\, dt$ $s.t.$ $y_t + Q(y_t, z_t, t) \geq y_{t+1}$ for all $t = 0, 1, \ldots, T$ $c \geq G(y_t, z_t, t)$ for all $t = 0, 1, \ldots, T$ $y(0)$ is given $y_{T+1} \geq y^*$	
Maximum Principle (Current Value)	
Current Value Hamiltonian	$H(y_t, \pi_{t+1}) = \max_{z_t} \beta^t F(y_t, z_t, t) + \pi_{t+1} Q(y_t, z_t, t)$ s.t. $c \geq G(y_t, z_t, t)$
FOC for z_t	$\frac{\partial H}{\partial z_t} = 0$ for all $t = 0, 1, \ldots, T$
Envelope Thm. for π_t	$\pi_{t+1} - \pi_t = -\frac{\partial H}{\partial y_t}$ for all $t = 1, \ldots, T$
Envelope Thm. for k_t	$k_{t+1} - k_t = -\frac{\partial H}{\partial \pi_{t+1}}$ for all $t = 0, 1, \ldots, T$
Initial condition	y_0 is given
Transversality condition	$\pi_{T+1}[y_{T+1} - y^*] = 0$
Maximum Principle (Present Value)	
Present Value Hamiltonian	$H(y_t, \pi_{t+1}) = \max_{z_t} \beta^t [F(y_t, z_t, t) + \pi_{t+1} Q(y_t, z_t, t)]$ s.t. $c \geq G(y_t, z_t, t)$
FOC for z_t	$\frac{\partial H}{\partial z_t} = 0$ for all $t = 0, 1, \ldots, T$
Envelope Thm. for π_t	$\beta^t \pi_{t+1} - \beta^{t-1} \pi_t = -\frac{\partial H}{\partial y_t}$ for all $t = 1, \ldots, T$
Envelope Thm. for k_t	$k_{t+1} - k_t = -\frac{\partial H}{\partial \beta^t \pi_{t+1}}$ for all $t = 0, 1, \ldots, T$
Initial condition	y_0 is given
Transversality condition	$\beta^T \pi_{T+1}[y_{T+1} - y^*] = 0$

Continuous Dynamic Maximization problem	
$$\max_{\{z(t)\}_{t=0}^{T},\{y(t)\}_{t=0}^{T}} \int_{t=0}^{T} e^{-\rho t} F\left(y\left(t\right), z\left(t\right), t\right) dt$$ $s.t.$ $Q\left(y\left(t\right), z\left(t\right), t\right) \geq y\left(t\right)$ for all $t \in [0, T]$ $c \geq G\left(y_t, z_t, t\right)$ for all $t \in [0, T]$ $y\left(0\right)$ is given $y\left(T\right) \geq y^{*}$	
Maximum Principle (Current Value)	
Current Value Hamiltonian	$H\left(y\left(t\right), \pi\left(t\right)\right)$ $= \max_{z(t)} e^{-\rho t} F\left(y\left(t\right), z\left(t\right), t\right) + \pi\left(t\right) Q\left(y\left(t\right), z\left(t\right), t\right)$ s.t. $c \geq G\left(y\left(t\right), z\left(t\right), t\right)$
FOC for $z\left(t\right)$	$\frac{\partial H}{\partial z\left(t\right)} = 0$ for all $t \in [0, T]$
Envelope Thm. for $\pi\left(t\right)$	$\dot{\pi}\left(t\right) = -\frac{\partial H}{\partial y\left(t\right)}$ for all $t \in [0, T]$
Envelope Thm. for $k\left(t\right)$	$\dot{k}\left(t\right) = -\frac{\partial H}{\partial \pi\left(t\right)}$ for all $t \in [0, T]$
Initial condition	y_0 is given
Transversality condition	$\pi\left(T\right)\left[y\left(T\right) - y^{*}\right] = 0$
Maximum Principle (Present Value)	
Present Value Hamiltonian	$H\left(y\left(t\right), \pi\left(t\right)\right)$ $= \max_{z(t)} e^{-\rho t} \left\{F\left(y\left(t\right), z\left(t\right), t\right) + \pi\left(t\right) Q\left(y\left(t\right), z\left(t\right), t\right)\right\}$ s.t. $c \geq G\left(y\left(t\right), z\left(t\right), t\right)$
FOC for $z\left(t\right)$	$\frac{\partial H}{\partial z\left(t\right)} = 0$
Envelope Thm. for $\pi\left(t\right)$	$\frac{de^{-\rho t}\pi\left(t\right)}{dt} = -\frac{\partial H}{\partial y\left(t\right)}$
Envelope Thm. for $k\left(t\right)$	$\dot{k}\left(t\right) = -\frac{\partial H}{\partial e^{-\rho t}\pi\left(t\right)}$
Initial condition	y_0 is given
Transversality condition	$e^{-\rho t}\pi\left(T\right)\left[y\left(T\right) - y^{*}\right] = 0$

Dynamic Optimization

Discrete Dynamic Maximization problem with Perfect Foresight	
$\max_{\{z(t)\}_{t=0}^{\infty},\{y(t)\}_{t=1}^{\infty}} \sum_{t=0}^{T} \beta^t F\left(y\left(t\right), z\left(t\right), t\right) dt$ $s.t.$ $y_t + Q\left(y_t, z_t, t\right) \geq y_{t+1}$ for all $t = 0, 1, \ldots$ $c \geq G\left(y_t, z_t, t\right)$ for all $t = 0, 1, \ldots$ $y\left(0\right)$ is given	
Dynamic Programming	
Bellman Equation	$v\left(y_t, t\right)$ $= \max_{z_t} F\left(y_t, z_t, t\right) + \beta v\left(y_{t+1}, t+1\right)$ s.t. $c \geq G\left(y_t, z_t, t\right)$ $= \max_{z_t} F\left(y_t, z_t, t\right) + \beta v\left(y_t + Q\left(y_t, z_t, t\right), t+1\right) + \lambda_t \left[c - G\left(y_t, z_t, t\right)\right]$
FOC for z_t	$F_z + \beta v' Q_z - \lambda_t G_z = 0$ for all $t = 0, 1, \ldots$
Envelope Thm. for y_t	$v' = F_y + \beta v' \left(1 + Q_y\right) - \lambda_t G_y$ for all $t = 1, \ldots$
Binding constraint	$y_t + Q\left(y_t, z_t, t\right) = y_{t+1}$ for all $t = 0, 1, \ldots$
Comp. Slackness	$\lambda_t \left[c - G\left(y_t, z_t, t\right)\right] = 0$ for all $t = 0, 1, \ldots$

Stochasitc Discrete Dynamic Maximization problem	
$\max_{\{z(t)\}_{t=0}^{\infty},\{y(t)\}_{t=1}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t F\left(y\left(t\right), z\left(t\right), t\right) dt$ $s.t.$ $\varepsilon_{t+1} = \rho \varepsilon_t + \eta_{t+1}$ for all $t = 0, 1, \ldots$ $y_t + Q\left(y_t, z_t, \varepsilon_{t+1}, t\right) \geq y_{t+1}$ for all $t = 0, 1, \ldots$ $y\left(0\right)$ is given	
Dynamic Programming	
Bellman Equation	$v\left(y_t, \varepsilon_t, t\right)$ $= \max_{z_t} F\left(y_t, z_t, t\right) + \beta E_t \left[v\left(y_{t+1}, \varepsilon_{t+1}, t+1\right)\right]$ $= \max_{z_t} F\left(y_t, z_t, t\right) + \beta E_t \left[v\left(y_t + Q\left(y_t, z_t, \varepsilon_{t+1}, t\right), \varepsilon_{t+1}, t+1\right)\right]$
FOC for z_t	$F_z + \beta E_t \left[v' Q_z\right] = 0$ for all $t = 0, 1, \ldots, T$
Envelope Thm. for y_t	$v' = F_y + \beta v' \left(1 + Q_y\right)$ for all $t = 1, \ldots, T$
Binding constraint	$y_t + Q\left(y_t, z_t, \varepsilon_{t+1}, t\right) = y_{t+1}$ for all $t = 0, 1, \ldots, T$

Appendix B

Appendix II EC751

Afterword

Acknowledgements

Bibliography