Repeated Bargaining

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ABSTRACT

This paper takes the noncooperative approach to study repeated bilateral bargaining problems. Extending traditional bargaining models, we allow two players to bargain for a number of times. Each bargaining opportunity is called a bargaining stage. In each stage, the bargaining is conducted according to the alternating-offer procedure and the stage game will end if consensus or deadline is reached.

By the number of stage and the number of period until deadline, we will analyze four cases separately: (i) finite-period, finite-stage bargaining; (ii) finite-period, infinite-stage bargaining; (iii) infinite-period, finite-stage bargaining and (iv) infinite-period, infinite-stage bargaining. We have found the three results: (i) when the number of stage is finite, subgame perfect equilibrium outcome is unique; (ii) For finite stage models, if the number of period is odd, the first mover must have advantage. However, if the number of period is even, it is the second mover having the advantage if both players are patient enough; (iii) using the concept of global stability, we could reduce the set of subgame perfect equilibrium for infinite stage models.

When we are to relax the convexity restriction on the bargaining set, we find that in infinite-stage models, we can induce long-run equilibrium outcomes that do not belong to the feasible set in stage game. To further investigate the implication, we will analyze the cases where bargainers having risk loving attitude and indomitable character. The former would retain the uniqueness of perfect equilibrium while the latter would not. Although, in the latter case, the multiplicity of equilibria is rather uninteresting, there are two interesting results if we restrict our attention to stationary strategy: (i) the set of stationary strategy SPE is finite; (ii) in asymmetric stationary SPE, first mover advantage would not disappear even when players are extremely patient.

摘要

本文以非合作賽局理論探討重迭雙人討價還價模型。作爲傳統討價還價模型的延伸，兩位議價者會有多次議價的機會。我們會把每一次的議價機會稱爲議價階段：在每一個議價階段，討價還價是以輪流提案的規則進行。每一個議價階段可設有時間限制，若討論時間超過限時，則該階段會自動結束。若沒有時間限制，則該階段會持續直到得到共識爲止。

根據議價階段的數目及每階段最多提議次數，本文所研究的模型將會分爲四種：(一)有限議價階段及有時間限制、(二)無限議價階段及沒有時間限制、(三)有限議價階段及沒有時間限制和(四)無限議價階段及有時間限制。本文得到三個結論：(一)有限議價階段的模型只存在唯一一個子賽局完全均衡、(二)在沒有時限的議價模型，若最多提議次數爲奇數，先行者必定佔優；但是若最多提議次數爲偶數，只要雙方有足夠耐性，先行者不一定有優勢，後行者反而可能佔優、及(三)對於無限議價階段的模型，若採用全局穩定的規範，則能縮少子賽局完全均衡集合。

我們發現若放寬可行集合的凸性假設，對於無限議價階段的模型中，可以達致可行範圍之外的結果並且能滿足子賽局完全均衡的要求。爲了深入研究箇中的因由，本文會分析喜好風險的議價者及不屈的議價者的議價行爲。前者會保留賽局完全均衡的唯一性，後者則存在無限個賽局完全均衡。雖然後者的推論有多重性，但是若考慮固定均衡，則其數目是有限的，而且其非對稱均衡的先行者優勢並不隨議價者的耐性增強而趨向零。

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While working through the project, he has guided me through the hurdles and painted the picture when I am losing in the maze. There are many times when he led me through the land of confusion and frustration. Besides technical problems, he has helped me in getting the thesis in better shape. From working with LaTeXto presentation of ideas, he has provided pieces and pieces of useful advice. Honestly, without his helpful guidance, I am afraid that I could not possibly complete this thesis. Needless to say, the remaining errors are all mine.

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1 Introduction

Bargaining is a struggle between cooperation and competition. Two parties must cooperate in order to produce surplus, or a "cake", as they could not do so by themselves alone. However, they have competition over the share of the surplus because in most of the cases more for one player would mean less for another player. If they can increase their own share without hurt the other side at the same time, what and why they are bargaining? Hence, two bargaining parties must have common interest and conflicting interest at the same time.

Traditional bargaining literature focuses on what division should be (as in axiomatic approach) and how the outcome of negotiation is reached (as in strategic approach). Under strategic approach, bargaining is usually modeled as a one-shot, extensive form game. To model long-term cooperation, one could simply model the whole period interaction as one game. Then, the surplus would represent the total benefit of whole period. In this sense, the sharing scheme which has been agreed at the beginning is fixed and cannot change even if both sides want to make amendment. Hence, the partition of surplus is rigid and not changeable during the whole period. Moreover, it would be hardly optimal for two sides to sign a long-term contract even if they know every single detail of each other. Usually, the environmental changes, implementation[1] and changes of preference might prove the cooperation is not beneficial to both sides. Therefore, such approach might not adequately include important elements such as renegotiation, intertemporal tradeoff and dynamic interaction.

This might explain why international trade negotiation is usually not completed in one setting but in phases. The best example is the Ministerial Conference of World Trade

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[1] Contracts, especially of long-term, might need some adjustments as implementation usually reflect inadequacies in earlier conception process.

Organization held every two years. Twenty-four months is enough for a great change in the business and political environment so that a new round of negotiation is needed as some infeasible agreements in the past might be feasible now and some agreements made in the past might not be suitable to continue. A local example would be the mainland and Hong Kong closer economic partnership arrangement (CEPA). The negotiation of the agreement is not done in one setting but in several phases. The first agreement was reached in 2003 and the subsequent talks are held while the liberation measure of previous phases are at work. Initial implementation are those business sectors without huge obstacles and later phases are for more complicated and technical industries. In this way, both sides could learn experience during the implementation, and develop trust and experience to work with other parties on issues with greater conflict. Another famous example is United States annual consideration to grant the most favor nation (MFN) status to China in 1990s . Although two countries have not directly discuss the matter on the table directly, both governments actually did undertake a undertable bargaining and an implicit bargaining where various performances of Chinese government including democratic progress are taken into the consideration of the granting of MFN status. This kind of annually revision of cooperation status is protecting interest of the strong side like a protecting clause in a contract signed between a strong producer and a weak seller because the strong side can unilaterally cancel the contract but no the weak side.

Election system in fact is a negotiation and renegotation process, which could be modeled as repeated bargaining. If we view the election process as a bargaining game between the political party and general public, democractic multi-party system is indeed a repeated bilateral bargaining with outside option of selecting other parties. One of the reasons for presidential and congressional elections to be held every few years is to ensure no one could continue to grasp the political power without continuous approval from the people of general public so that powerless individuals are protected against any unjustified action from powerful government. In the same vein, corporation with many shareholders would have the rights to choose the members of board of directors to represent their interest in the company.

We could hardly incorporate these interactive elements unless we are to model the long-term interaction as repeated game. With future cooperation opportunities, players might make strategic concession today in return for future concession from other parties tomorrow. Making commitments, tax cut and lax welfare policy are common tactics used by the ruling party during the election year. Although it might be unwise to adopt such strategy if we only consider the present payoff, winning the election would mean the total payoff increased and hence it might be wise after all. Therefore, concept of efficiency would be completely different as players would view the whole sequence of bargaining rather than thinking continuation of bargaining under certain probability.[2]

Another important implication is that outcomes which are infeasible in one-shot bargaining could be achieved in multi-stage bargaining when the restriction of convex feasible set is relaxed. In real-life negotiation, there are many occasions in which the feasible set is non-convex. In student dormitory, each roommate would be responsible for cleanliness of the room shared with roommates. From standard bargaining theory, we would predict each student would have do his own task each day.[3] However, we would rarely see this kind of arrangement. Instead of sharing of daily tasks, students would take turn to complete the whole task each day partly because it is more efficient in terms of actually doing the job and monitoring of the task.[4] Borrowing terminologies from theory of production, the

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[2] The continuation conception is a classical view of one-shot game. The discount factor is treated to be the combination of continuation probability and degree player's patience.

[3] One might suggest that the surplus could be consider as whole period gain such that each one doing one day is the prediction from theory. However, if we are to interpret this way, we would assume the player has entered a long-term contract without any chance of withdrawal.

[4] Isn't it easier to check a job has been done by person in charge than part of the task has been completed

fixed cost of doing the simple duty is quite high compared to variable cost. Then, it is not optimal for all of the roommates to do the task in parts at the same time. If each one alternatively does all the tasks that day, the cost would be lowest for all. Hence, with the increasing marginal return to scale, the feasible set is non-convex such that by division of labor across time, students could achieve the agreement which is not feasible if they are to cooperate once only.[5] This kind of efficient argument could also be applied to give an explanation to the division of labour between each functional department in modern corporation.

Our model would be built on the foundation of alternating-offer bargaining model by Stahl(1972) and Rubinstein(1982). As strategic models, they allow each player make a proposal to another player who could accept or reject it. Acceptance would end the game and the surplus would be divided according to the agreement while rejection would allow the other party to make a counter-offer to be proposed after certain period of time. The only difference is that the former imposes exogenous pre-determined deadline by which the game ends automatically if there is no agreement reached and the latter allows bargaining to continue forever unless consensus is being reached. We call such bargaining game as a bargaining stage. Following Muthoo (1995), our models of repeated bargaining would simply infinitely repeatedly playing the bargaining stage but with one special adjustment. Except the very first stage where one player is picked as the first mover, the first proposer of each stage would be the one who have accepted the offer from other player in the previous stage.

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by the one responsible?

[5] In the first presentation of this paper, Prof. Kwong has pointed out the problem why players do not wish to draw lottery but to bargain over the work. He has suggested it might be due to mechanism design problem. My point of view is that ex ante efficiency and ex post efficiency should be treated differently in one-shot game and in repeated game. In one-shot game, the former might be more important. However, in repeated game, the latter is more important given players might regret and refuse to follow the lottery outcome.

Combining the idea from Stahl, Rubinstein and Muthoo, we established a family of repeated bargaining models with two parameters: the number of period in each bargaining stage, denoted by z and the number of bargaining stage, denoted by n. By finiteness of the parameters, we would analyze four models: (i) finite-period, finite-stage bargaining $G(n, z)$; (ii) finite-period, infinite-stage bargaining $G(n, \infty)$; (iii) infinite-period, finite-stage bargaining $G(\infty, z)$ and (iv) infinite-period, infinite-stage bargaining $G(\infty, \infty)$.

For models with deadline ($z < \infty$), surprisingly, as long as both players know when their relationship must end ($n < \infty$), the set of subgame perfect equilibria contains only one element. One result is that the first mover might not be able enjoy the advantage of being the first proposer when number of stage n is a odd even number or when size of surpluses are larger in latter stages. In such cases, the last mover does have the advantage as he could issue ultimatum in the last round which compels the other player to succumb to tougher terms. Another result is that the equilibrium offers decrease as the stage game approach deadline and as the remaining number of stage decreases. The reason for the decrease as deadline approaching is that the ultimatum effect becomes more evident and the reason for the drop as the number of stage decreases is that the cost of delay of future agreements due to rejection drops. When there is no predetermined date for termination of relationship ($n = \infty$), the game could not be solved by backward induction and the game would actually become infinitely repeated game and such game allows almost any outcome to be supported as subgame perfect equilibrium (SPE) due to the fact that the folk theorem applies.

For models without deadline ($z = \infty$), if the relationship is to terminate after one cake is divided ($n = 1$), we would have unique SPE as Rubinstein proved in his seminal paper [5]. In the case of more than one cake are to be divided ($n > 1$), we would expect, due to folk theorem, the set of subgame perfect equilibrium would expand greatly. However, if there is a limit on the number of cakes to be divided ($n < \infty$), there exists a unique equilibrium with stage-stationary path which is equal to the limiting outcome of corresponding models with deadline. When there is no predetermined date for termination of relationship ($n = \infty$), there will be a unique stationary strategy SPE but infinitely many non-stationary SPEs. To refine the set of SPEs, we have applied global stability and successfully eliminated those extremal, counter-intuitive outcomes.

Although the multiplicity problem associated with non-convex bargaining reduce the predicative power of the deduction, we will try to analyze two simple situations which are common in practice. The first case is that bargainers have increasing marginal utility or they are are risk-lovers. To model the increasing return, one could not count on the traditional bargaining model but have to adopt the behavorial assumption. The second case is an "upgraded" version of the first one: players are indomitable in the sense that the bargainers are eager to get more than 50% of the surplus.[6] Usually, even when we know that at the weak position, few of us would be happy that we are getting less than one-half of the surplus. Even in ultimatum game[7], player would reject offer of 30% because "they would rather forgo some money than be treated unfair".

This thesis is organized as follows. Section 2 provides a detailed literature review. Section 3 will discuss the model setup. Section 4 and section 5 will characterize the equilibrium for finite stage model and infinite stage model respectively. Section 6 would discuss the two special non-convex examples. Section 7 investigates the application of various models. Section 8 contains a summary and concluding remarks.

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[6] I must admit that the present model might not capture all the dynamics of indomitability. One important component has ignored in bargaining model is the implementation problem. Without perfect monitoring, if one is getting less, he might make less effort in the cooperation phases. Though such kind of behavior would have no effect on one-shot bargaining, they do have important implication if both parties are to cooperate in the future.

[7] See [14] for a detailed meta-analysis on the experimental results in ultimatum game.

2 Literature Review

Although bargaining is one of important component in transaction process,[8] economists have said very little about it until the recent 60 years. It was perhaps theorists lacked of analytical skills to develop models for agents with conflict of interest. Interactive decision theory, probably except Cornot, was not available for solving the dynamic situation inherent in bargaining before 1950s. One may, therefore, suggest that the development of bargaining theory should be highly correlated to the development of game theory and indeed both two theories made great progress at the same time.

The first formal work in bargaining theory probably could be traced to Nash. He solved the problem using two approaches: (i) cooperative axiomatic approach and (ii) noncooperative strategic approach. In the first approach [19], he characterized the unique solution by four axioms. The second approach [20] is the so-called Nash demand game where each player simultaneously making offers with threat. Nash believed the two approaches were complementary to each other and he had implemented the axiomatic solution by using the strategic approach.[9]

The development of axiomatic approach mainly focuses on the selection of more general and less restrictive axioms. Particularly, due to problems with the axiom of independent of irrelevant alternatives, various alternative solution concepts are proposed. The most popular two alternatives are egalitarian solution [11] and Kalai-Smorodinsky solution [12]. Of course, three of them are similar solution concepts and they only differ in the choice of set of axioms[10]. For further discussion on them, refer to [28].

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[8] Other components can be searching and matching.

[9] Though Nash acknowledged the importance of implementation of the axiomatic solution by strategic setup, the setup of Nash demand game is too artificial and fails to incorporate the essential dynamic ingredient in bargaining. It would not be surprising to expect that after the seminal work by Rubinstein, numerous works on implementation come out. Binomore, et. al [21] implements the nash solution using the alternating-offer bargaining procedure and theories on implementing various setting of bargaining models are still in active research.

[10] Nash solution requires the equilibrium distribution to be weakly better than other options and adopts

Even though Nash demand game is artificial and unnatural, there is not much development in strategic approach until 1970s, though extensive form game has been widely available during 1950s. Stahl developed the first strategic finite-horizon bargaining model using backward induction in 1970s and Rubinstein has developed the infinite-horizon counterpart in 1980s. Most of subsequent papers adopted Rubinstein's model as basic skeleton. One of the most remarkable result of Rubinstein model is the uniqueness of perfect equilibrium. Unfortunately, the uniqueness property seems sensitive to the assumptions. For example, when van Damme, Selten and Winter [13] and Muthoo [1] breaks down the continuity in the feasible set, every possible agreement can be support as SPE if players are patient enough or time lag between offers and counteroffers are short enough. Binmore [22] extended the analysis of Rubinstein to general feasible set. He has applied the concept of iterated deletion of conditional dominated strategy to characterize the set of containing SPE. Until now, for non-convex bargaining, there is not much development in strategic approach but there are quite a few paper in axiomatic approach.

As an extension to standard bargaining model, possibly Fershtman [8] was the first to consider multi-issue[11] bargaining though bargainers would cease their relationship after agreements on several issues reached. He considered three cases: (i) payoff is obtained immediately after an agreement is reached; (ii) payoff is realized only after agreements on all issues in agenda is reached; and (iii) stream of payoff could be obtained after an agreement is reached but would cease if there is no final consensus in all issues in the

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axioms of efficiency, symmetry, scale invariance and contraction independence. Egalitarian solution requires the distribution to be equal among players and uses axioms of weak efficiency, strong symmetry and contraction independence. Kalai-Smorodinsky solution requires the ratio of ideal point to the equilibrium distribution to be equal among players and uses axioms of weak efficiency, strong symmetry, scale invariance and weak contraction independence.

[11] In theory, multi-issue bargaining and repeated bargaining are same except the realization of payoff and bargaining opportunity. In most multi-issue bargaining models, the payoff is realized only after all issues on the agenda has reached and all bargaining opportunities are known. In repeated bargaining, the payoff is realized immediately but the bargaining opportunity only appears after the previous bargaining problem is settled.

agenda. He claimed that only the second cases is the only interesting case to analyzed as the first case is just a trivial extension as each issue could be treated independently and the result of third case is similar to second one. Later, Muthoo [2] worked on the first case with a little change on how the repetition is done and found out the result is not so trivial. In his model, the time lag between offers and counteroffer in bargaining might not be the same as the time lag between conclusion of one bargaining and initiation of the new one. In effect, he has extended the Rubinstein model by infinitely repeating the alternating-offer bargaining model and the result obtained is different from the standard Rubinstein solution because bargaining across stages are interdependent. Since the model is in fact a repeated game, he has found out that the set of non-stationary equilibria indeed include all the outcomes in feasible set, which is similar to the result from folk theorem.

3 Model

Denote the two bargainers by player A and player B. Two players would have opportunity to enter n bargaining situations.[12] We call each bargaining situation a bargaining stage. Each bargaining stage lasts for z periods.[13] Hence, we can describe the family of repeated bargaining model by $G(n,z)$ using two parameters: (i) $G(n,z)$ finite stage, finite period model ; (ii) $G(n,\infty)$ finite stage, infinite period model; (iii) $G(\infty,z)$ infinite stage, finite period model and (iv) $G(\infty,\infty)$ infinite stage, infinite period model. We will discuss each models according to the following schedule in shown Table 1.

Stage\Period	Finite	Infinite
Finite	Section 4, I	Section 4, II
Infinite	Section 5, II	Section 5, III

Table 1: Schedule of discussion of four models

Each stage proceeds according to the alternating offer procedure. Except the very first bargaining stage where player A is assumed to be the first proposer, if player A accepts player B's offer in the previous stage, player A would be the first mover next stage, and vice versa.[14] If a player accepts other's proposal, the present bargaining stage (kth stage)

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[12] Here, the opportunity is used because for bargaining without deadline, the stage game would no end if no consensus is reached. The number of bargaining stages n might be finite or infinite. When we say n is finite, we mean that each bargainer would expect their cooperative relationship ends after n agreements are reached. For infinite n, we mean each bargainer would expect not think their relationship terminate in foreseeable future

[13] Here, we assume the stage is same across stages and its duration is counted in discrete period of time. This setting is enough for most practical purpose.

Each bargaining stage could last for finite time (finite z, or bargaining with deadline) or infinite time (infinite z, or bargaining without deadline). When we model the situation as finite horizon bargaining, it means that both players have clear idea that bargaining would end in the foreseeable future if they do not reach agreement soon enough. There are many cases bargaining would end in a certain period of time, especially in most political bargaining. When we adopt infinite horizon bargaining model, it means that both bargainers expect bargaining would not end in foreseeable future or they do not think the bargaining opportunity would end before they reach the agreement.

[14] This setup follows Muthoo [2]. There is a little subtlety when rejection appear in the last round for finite horizon bargaining because there are two possible way to assign who is the first proposer in next stage. One is to allow the last proposer continue to be first proposer. Another one is to let the one who rejects the last proposer to be the first proposer. We shall see that both cases would return similar outcome

ends and the next stage ($(k+1)$th stage) would start after time τ[15] as long as the final stage has not been reached, that is $k+1 \leq n$. However, if the player rejects the offer, after time Δ, he could make an counteroffer. Hence, both players are sequentially making offers and counteroffers to each other with time lag Δ until either one agrees with other's proposal. After an agreement is reached, the next stage would start after time τ.

Formally, each stage itself is a bargaining problem and we denote the kth stage bargaining problem by $X_k \cup \{d_k\}$ where X_k be the set of feasible agreements and d_k be disagreement outcome in kth bargaining stage.[16] Usually, we would assume there is underlying conflict of interest between players, that is, given a sharing scheme, if player A has to increase his own share, player B must decrease eventually, and vice versa. Otherwise, both player could get infinite amount surplus. One consequence of this assumption is that when player A is getting the highest possible share of surplus in kth stage, player B is getting the lowest possible share of surplus in that stage.

Repeated bargaining model $G(n,z)$ is n stages bargaining problems joined together sequentially and we denote it by $\{X_k \cup \{d_k\}\}_{k=1}^{n}$. To simplify the problem, each possible agreement is represented by a 2-tuple $(x_k, y_k) \in X_k$ and the perpetual disagreement outcome is represented by $d_k = (d_A, d_B)$. If a k-th stage agreement (x_k, y_k) is reached at time T_k (k is counted from one), the outcome of this stage is denoted as $\langle (x_k, y_k), T_k \rangle$. Hence, denote

$$\langle (x_1, y_1), T_1, \ldots, (x_n, y_n), T_n \rangle \in \times_{k=1}^{n} \{X_k \times [0,\infty) \cup (d, \infty)\}$$

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in equilibrium if the bargaining period is odd. However, in even period bargaining, the latter assumption might lead to strategic delay. In most cases, we will analyze the models under the former assumption as it would be much easier. See appendix I for discussion of this assumption and strategic delay.

[15] This might represent the time required for both parties to consume the old cake, time required for new opportunity to come and time required to come up with new agreement,

[16] In most paper in the literature, the author simply stated the feasible set is a convex set without stating clearly the forms of utility or preference. However, as we wish to break the assumption of convexity, it would be better to start from the very beginning.

to be the bargaining outcome of n-stage bargaining problem. If we denote $t_k \in [0, z\Delta]$ be the time passed from the beginning of kth stage to the kth stage agreement reached, then we have $T_{k+1} = T_k + t_k + \tau = \sum_{i=1}^{k} t_i + (k-1)\tau$. Therefore, when we say the offer is make in t_k/Δth period in kth stage, we would denote it as $\left(x_k^{t_k/\Delta}, y_k^{t'_k\Delta}\right)$. Note that t_k must be multiple of Δ and T_k must be sum of multiple of Δ and multiple of τ.

Assume players' preferences $\succsim_i^k$ $(i = A, B)$ satisfy the axioms of complete, transitive, independence, continuity and time-invariant. Then, a vNM utility function could be used to represent the preference over the outcome space $\{X_k \cup \{d_k\}\}_{k=1}^{n}$. As the preference would be extremely difficult to deal with directly in repeated bargaining, we would assume the existense of utility function (U for player A and V for player B) over n bargaining stage and the function is separable additively,[17] that is,

$$\begin{aligned} U\left(\langle(x_1, y_1), T_1 \ldots, (x_n, y_n), T_n\rangle\right) &= U\left(\langle(x_1, y_1), T_1\rangle\right) + \ldots + U\left(\langle(x_n, y_n), T_n\rangle\right) \text{ and} \\ V\left(\langle(x_1, y_1), T_1 \ldots, (x_n, y_n), T_n\rangle\right) &= V\left(\langle(x_1, y_1), T_1\rangle\right) + \ldots + V\left(\langle(x_n, y_n), T_n\rangle\right) \end{aligned}$$

If sum of share of surplus of two players is constant, we could assume each player care his own payoff only. Moreover, we also assume separability between effect of time preference and outcome preference[18] and players discount future payoff. Denote $r_i > 0$ as player i's rate of time preference. Hence, we have

$$\begin{aligned} U\left(\langle(x_k, y_k), T_k\rangle\right) &= U(x_k)\exp(-r_i T_k) = u_k^{T_k} \text{ and} \\ V\left(\langle(x_k, y_k), T_k\rangle\right) &= V(y_k)\exp(-r_i T_k) = v_k^{T_k}. \end{aligned}$$

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[17] Usually, getting more or less in one round would not affect the preference in subsequent rounds. Each bargaining outcome is independent of each other which means players treat previous outcome "sunk". This assumption is particularly fit for the case in which time between bargaining rounds are long or bargainers would immediate enjoy the payoff after agreement is reached.

[18] See [27] for the detailed discussion on the condition of the validity of the separability of time and outcome preference.

To save notation, let $\delta_i = \exp(-r_i\Delta)$ and $\alpha_i = \exp(-r_i\tau)$ for $i = A, B$. Denote $\mathcal{U}_k^{T_k}$ the possible utility pairs in stage k and at time T_k, that is,[19]

$$\mathcal{U}_k^{T_k} = \left\{ \left(u_k^{T_k}, v_k^{T_k}\right) : \left(U^{-1}\left(u_k^{T_k}/\delta_A^{T_k}\right), V^{-1}\left(v_k^{T_k}/\delta_B^{T_k}\right)\right) \in X_k \right\}.$$

Pareto frontier $\Omega_k^{T_k}$ in stage k and at time T_k would then be

$$\Omega_k^{T_k} = \left\{ \left(u_k^{T_k}, v_k^{T_k}\right) : \text{there does not exist } (a,b) \in \mathcal{U}_k^{T_k} \text{ such that } a > u_k^{T_k} \text{ and } b > v_k^{T_k} \right\}.$$

Sometimes we might need to consider the payoff from the whole game, not just from stage game only. Define $\tilde{T} = (T_1, \ldots, T_n)$ be the sequence of time for n agreements reached and $\tilde{T}_k = (T_k, \ldots, T_n)$ be the subsequence of time for kth stage agreements to nth stage agreement reached which is a subsequence of $\tilde{T}$. Similarly, we would let $\Omega_k^{\tilde{T}_k}$ be the set of cumulative Pareto Frontiers

$$\begin{aligned} \Omega_k^{\tilde{T}_k} &= \sum_{h \in \{k,..,n\}, T_h \in \tilde{T}_k} \Omega_k^{T_k} \\ &= \left(\sum_{h \in \{k,..,n\}, T_h \in \tilde{T}_k} u_h^{T_h}, \sum_{h \in \{k,..,n\}, T_h \in \tilde{T}_k} v_h^{T_h} \right). \end{aligned}$$

Similar to most papers in the literature, we would focus on pure strategy as it is hardly plausible for players to use randomization device in any practical bargaining situation. As in most paper in the literature, we would focus on pure strategy. A strategy is stationary if the offering rule and acceptance rule is history-independent, and the same within stage and across stages. A strategy is stage-stationary if he offering rule and acceptance rule is history-independent, and the same in each stage but different across stages. A strategy

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[19] Note that for $u_k^{T_k}$ and $v_k^{T_k}$, the subscript is to denote the time when the agreement is reached but not the power to be raised.

is stage-period-stationary if he offering rule and acceptance rule is history-independent but different according to stage and period. Otherwise, we call the strategy to be non-stationary. Throughout this paper, we would adopt the assumption that there exists bijection between payoff and strategy.

4 Equilibrium for finite period bargaining

I Finite stage bargaining model: $G(n, z)$

Firstly, we would like to show that under the certain restriction on the set of Pareto Frontiers $\Omega_k^{T_k}$,[20] we would have a unique SPE for finite stage bargaining. Surprisingly, unlike infinite period models, uniqueness retains even if the feasible set (and hence the Pareto frontier) consists only finite number of points.[21] One special feature of equilibrium of finite stage finite period model is that equilibrium strategy would depend only on stage and period of the bargaining but independent of history, which means that any deviation by mistake would not lead to off-equilibrium path but delay only.

Proposition 1 *For bargaining problem $G(n, z)$ with $\{X_k \cup \{d_k\}\}_{k=1}^{n}$ and preference ordering $\left\{\succsim_A^t, \succsim_B^t\right\}$, the game has a SPE with immediate agreement*[22] *if and only if the set of Pareto Frontiers $\left\{\Omega_k^{T_k}\right\}_{k=1}^{n}$ of satisfy the condition that for each element in $\Omega_k^{\tilde{T}'_{k'}}$, there exists at least one Pareto superior element in $\Omega_k^{\tilde{T}_k}$ for $k \leq k'$, $\tilde{T}_k \leq \tilde{T}'_{k'}$. Furthermore, if for all $T_k \leq T'_k$, every element in $\Omega_k^{T_k}$ are strictly dominated by some elements on $\Omega_k^{T'_k}$, SPE is unique and its strategy is stage-period stationary.*

Proof. To ensure immediate agreement, we must have $\tilde{T}_k \leq \tilde{T}'_k$ for all possible $\tilde{T}'_k$ such that all elements in $\Omega_k^{\tilde{T}_k}$ must not be Pareto dominated by any elements in $\Omega_k^{\tilde{T}'_k}$. Otherwise player would simply wait and get higher payoff. Then, it is clear that existence of Pareto superior element is sufficient condition for immediate agreement.

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[20] Note that the set of Pareto frontiers is finite as T_k assumes finite number of values. More precisely, there are $n \times z$ Pareto frontiers.

[21] In the paper by E.V, Damme, R. Seltan and E.Winter [13], they have shown that in standard Rubinstein bargaining game, any outcome (including outcomes with delay and even perpetual disagreement) could be supported as perfect equilibrium if players are patient enough.

[22] More precisely, we need that for all $k \leq k'$, and all $T_k \leq T'_k$, we have $\left(\Omega_k^{T_k} \cup \Omega_k^{T'_k}\right) \subset \Omega_k^{T_k}$

For immediate agreement, it means that there is no delay in each round, that is $\tilde{T}_k = ((k-1)\Delta, \ldots, (n-1)\Delta)$. Obviously, this implies that all elements in $\Omega_k^{\tilde{T}_k}$ must be some element not Pareto dominated by any element in $\Omega_k^{\tilde{T}'_k}$ for all $\tilde{T}_k \geq \tilde{T}'_k$. Therefore, the existence of equilibrium of immediate agreement would be necessary condition for the existence of Pareto superior element.

The construction of the equilibrium strategy is straightforward. As T_n is bounded above by $z\Delta + (n-1)\tau$, we could apply backward induction. Hence, it suffices to illustrate the case of a bargaining game with one stage and n periods as an induction argument using backward induction completes the proof for any finite many stages. Since stage game ends in time z, one can define

$$\Omega_1^{(z+1)\Delta} = \{U(d_A)\,\delta_A^z, V(d_B)\,\delta_B^z\} \equiv \Omega_1^t \text{ for all } t > (z+1)\,\Delta.$$

Hence, we only need to focus on Ω_1^t for $t \leq (n+1)\,\Delta$. Now consider when $t = z\Delta$, since $\Omega_1^{(z+1)\Delta}$ contains only disagreement payoff, we could always find outcomes on the Pareto frontier $\Omega_1^{z\Delta}$ such that payoff would be better than the disagreement. Among those outcomes, we could be able to find a outcome which would maximize the payoff of player making the last proposal (if z is even, then it is player A; otherwise, player B). The reason we focus on the frontier alone is because any outcome not on the frontier is not optimal for both players. The existence of maxima on the frontier is ensured by the finiteness of frontier (if it consists of discrete point) or compactness of frontier (if it is a connected line segment). Similarly, by the assumption of existence of Pareto superior outcome, when $t = (z-1)\,\Delta$, we could find a unique point on Pareto frontier $\Omega_1^{(z-1)\Delta}$. Repeating this process, we could find a unique point on Ω_1^0 which is the unique SPE outcome of the game. For uniqueness of equilibrium strategy, it would be ensured by the existence of bijection between payoff and strategy. As construction of equilibrium payoff is unique along the path of play, we know that the strategy must be stage-period stationary. ■

Carefully looking at the strategy, we could observe the following fact:

Remark 1 *For one stage bargaining model, $G(1,z)$, a bargainer being the last mover could obtain all the surplus. Therefore, proposer could have extra advantage if the number of period remaining is odd. We call such effect as last-mover advantage. Moreover, as players are impatient, being proposer could also have extra advantage over the other player in all cases. We call such effect the first-mover advantage.*

When the number of period of each stage is odd, the first proposer could enjoy the first-mover advantage and last-mover advantage at the same time. However, when the number of period of each stage is even, the first-mover advantage and last-mover advantage do not act on the same person. Table 2 summarizes the effects under different setting.

Number of Period	Even period (odd z)		Odd period (even z)	
Role of the player	proposer	respondent	proposer	respondent
first-mover advantage	+ve	-ve	+ve	-ve
last-mover advantage	-ve	+ve	+ve	-ve

Table 2: The qualitative effect of first-mover advantage and last-mover advantage under different settings

By looking at the patience and the number of period in each stage, we find out that the following relationship.

Remark 2 *Last-mover advantage is enhanced when the players are more patient and number of period in each stage reduces while the first-mover advantage is enhanced when the players are less patient and the number of period in each stage increases.*

(Therefore, for infinite period game, one may suggest that the last mover advantage approached to zero and only the first mover advantage is observed.)

In multi-stage game $G(n,z)$ where $n > 1$, if number of period in next stage is odd, respondent is willing to accept the lower offer than otherwise because he can become the first mover in the next stage where he can enjoy both first-mover advantage and last-mover advantage. If the number of period in next stage is even,the proposer is willing to offer more to the respondent if players are sufficiently patient or if surplus in the next stage is large enough. It is because the last-mover advantage will be greater than the first mover advantage so that the net effect is that being the first mover is worse than the second mover. We would like to call such these concession behaviours solely due to consideration of first-mover status as strategic concession. To look into the matter clearly, we would like to restrict our attention the standard bargaining problem where players are risk neutral and the sum of surplus is a constant.

Proposition 2 *For finite stage bargaining model,* $G(n,z)$*, suppose that the bargaining problem for each stage is standard bargaining problem, that is all* X_k *is standard unit simplex and all* d_k *are* $(0,0)$*. We also assumes that that players are risk netural,* $U(x) = x$ *and* $V(y) = y$*, and have same time preference,* $r_A = r_B = r$*.*[23] *Then the set of SPE has have the following properties:*

1. *the game has only a unique perfect equilibrium offers* x^*_{ij}[24]

2. *If number of period in each stage is even and holding* j *constant, equilibrium offers* x^*_{ij} *is non-increasing as* i *increases.*

3. *If number of period in each stage is odd and holding* j *constant, there will be a* r^* *such that equilibrium offers* x^*_{ij} *are constant as* i *increases, and for any* $r < r^*$*,* x^*_{ij} *is*

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[23] The reason for such assumptions is just to keep other factors constant. I believe these restrictions are not crucial and conjecture this result should hold for more general case but I lack the time and skill to complete the proof.

[24] We denote the equilibrium offers of proposer's share in stage i, period j to be $x^*_{ij} = x_i^j$ if it is player A makes proposal and $x^*_{ij} = y_i^j$ if it is player B makes the proposal.

non-decreasing as i increases, and for any $r > r^*$, x^*_{ij} *is non-increasing as i decreases.*

4. *If number of period in each stage is even and holding i constant, equilibrium offers* x^*_{ij} *would be fluctuating up and down in alternating manner. The magnitude of switch is decreasing expoentially with time and eventually approach to zero. If* $\tau - \Delta \geq 0$*, the offers* x^*_{ij} *would go up first and then go down. Otherwise, the switch would depend on players' time preference. There will be a* r^*_k *such that for any* $r > r^*_k$*, the offers* x^*_{ij} *go down and then go up, and for* $r < r^*_k$*, the offers* x^*_{ij} *go down and go up.*

5. *If number of period in each stage is even and holding i constant, there would be a* r_{Θ_k} *such that when* $r > r^*_{\Theta_k}$*, the result is same as (4) but when* $r < r^*_{\Theta_k}$*, all analysis in (4) would be reversed.*

Proof. (1) For the proposition 1, it can be seen that the perfect equilibrium outcome is unique.

For any game $G(n, z)$, the last $n-1$ stage games should be the same as $G(n-1, z)$. Therefore, $x^*_{n,z}$ would be the same for all $G(n-k, 1)$ for all $k = 0, 1, \ldots n-1$. First, using backward induction, one could easily shown that

$$x^*_{n,z-t} = 1 - \delta + \delta^2 - ... + (-1)^t \delta^t$$

Now, moving one stage backward, we are in the $(n-1)$th stage. In the last period this stage, since it is a ultimatum game, we have

$$x^*_{n-1,z} = 1.$$

Moving t periods backward, in the $(z-t)$th period, when one player is to make the offer $x^*_{n-1,z-1}$, equilibrium requires that respondent should be indifferent between acceptance and rejection:

$$1 - x^*_{n-1,z-t} + \alpha x^*_{n,0} = \delta x^*_{n-1,z-t+1} + \alpha\left(1 - x^*_{n,0}\right)$$

Hence, if we do recursive substitution, we would have

$$\begin{aligned}
x^*_{n-1,z-t} &= \left(1 - \delta x^*_{n-1,z-t+1}\right) + \alpha\left(2x^*_{n,0} - 1\right) \\
&= \left[1 - \delta\left(1 - \delta x^*_{n-1,z-t+2}\right)\right] + \alpha\left(1-\delta\right)\left(2x^*_{n,0} - 1\right) \\
&= \left[1 - \delta + \delta^2 - \delta^3 x^*_{n-1,z-t+3}\right] + \alpha\left(1 - \delta + \delta^2\right)\left(2x^*_{n,0} - 1\right) \\
&= \left[1 - \delta + \delta^2 - \delta^3 + \cdots + \left(-1\right)^t \delta^t x^*_{n-1,z}\right] \\
&\quad + \alpha\left(2x^*_{n,0} - 1\right)\left[1 - \delta + \delta^2 + \cdots + \left(-1\right)^{t-1}\delta^{t-1}\right] \\
&= x^*_{n,z-t} + \alpha\left(2x^*_{n,0} - 1\right) x^*_{n,z-t+1}.
\end{aligned}$$

When we are at the $(n-k)$th stage stage, we would have

$$x^*_{n-k,z-t} = x^*_{n,z-t} + \alpha\left(2x^*_{n-k+1,0} - 1\right) x^*_{n,z-t+1}$$

which means that

$$x^*_{n-k,z-t} - x^*_{n-k+1,z-t} = 2\alpha\left(x^*_{n-k+1,0} - x^*_{n-k+2,0}\right)$$

so that $x^*_{n-k,z-t} \geq x^*_{n-k+1,z-t}$ for all t if and only if $x^*_{n-k+1,0} \geq x^*_{n-k+2,0}$. It is clearly that the condition is recursive so that it could be easily satisfied inductively.Denote $P\left(k,t\right)$ be the proposition "$x^*_{n-k,t} \geq x^*_{n-k+1,t}$". When the inequality $P\left(k',0\right)$ is true for some k', then $P\left(k'+1,t\right)$ will be true which means $P\left(k'+1,0\right)$ is true as a special case. Then, principle of mathematical induction would automatically complete the proof for $k = k'+1, k'+2, \ldots$.

(2) We first consider the case of even z first. We know that

$$\begin{aligned} x^*_{n,0} &= 1-\delta+\delta^2-...+(-1)^z\delta^z \\ &= \frac{1+\delta^{z+1}}{1+\delta} \geq \frac{1}{1+\delta}. \end{aligned}$$

Hence,

$$\begin{aligned} 2x^*_{n,0}-1 &\geq \frac{2}{1+\delta}-1 \\ &= \frac{1-\delta}{1+\delta} > 0 \end{aligned}$$

so that we have $x^*_{n-1,z-t} \geq x^*_{n,z-t}$ for all t. As we have shown $P(0,0)$ is true, we could say that $x^*_{n-k,z-t} \geq x^*_{n-k+1,z-t}$ for all t and k. Hence, we have shown that the equilibrium offer x^*_{ij} is non increasing as i increases, holding j constant.

(3) Now consider the case of odd z. We know that

$$x^*_{n,0} = 1-\delta+\delta^2-...+(-1)^z\delta^z = \frac{1-\delta^z}{1+\delta}.$$

Hence,

$$\begin{aligned} 2x^*_{n,0}-1 &= \frac{2(1-\delta^z)}{1+\delta}-1 \\ &= \frac{1-\delta+2\delta^z}{1+\delta} \equiv k \end{aligned}$$

so that $x^*_{n-1,z-t} \geq x^*_{n,z-t}$ if and only if $k \geq 0$. One can easily observe

$$\begin{aligned} \frac{\partial k}{\partial \delta} \geq 0 &\Leftrightarrow 2z\delta^{z-1}-1 \geq 0 \\ &\Leftrightarrow \delta \geq \exp\left(\frac{\ln(1/2z)}{z-1}\right) \equiv \delta^* \in [0,1] \end{aligned}$$

It is clear that $\delta^* \in [0,1]$. Also, when $\delta = 0$, $k = 1$ and when $\delta = 1$, $k = 0$. With one stationary point at δ^*, we know that there exists one δ' such that $k = 0$ and for any $\delta > \delta'$, we have $k < 0$, and for any $\delta < \delta'$, we have $k > 0$. Using same induction logic in (2), the result can be easily shown.

Before we prove (4) & (5), it is useful to look at the trend of offers within a stage. First, we would look at the last stage, the general formula for the offers are:

$$x^*_{n,z-t} = \begin{cases} \dfrac{1+\delta^{t+1}}{1+\delta} & \text{if } t \text{ is even} \\ \dfrac{1-\delta^{t+1}}{1+\delta} & \text{if } t \text{ is odd} \end{cases}$$

Hence,

$$x^*_{n,z-t} - x^*_{n,z-t+1} = \begin{cases} \delta^t & \text{if } t \text{ is even} \\ -\delta^t & \text{if } t \text{ is odd} \end{cases}$$

Now consider the case when we are in the $(z-t)$th period of kth stage, we have shown that

$$x^*_{n-k,z-t} = x^*_{n,z-t} + \alpha\left(2x^*_{n-k+1,0} - 1\right) x^*_{n,z-t+1}$$

which means that in the $(z-t+1)$th period of the same stage,

$$x^*_{n-k,z-t+1} = x^*_{n,z-t+1} + \alpha\left(2x^*_{n-k+1,0} - 1\right) x^*_{n,z-t+2}$$

so that

$$\begin{aligned}
x^*_{n-k,z-t} - x^*_{n-k,z-t+1} &= \left(x^*_{n,z-t} - x^*_{n,z-t+1}\right) + \alpha\left(2x^*_{n-k+1,0} - 1\right)\left(x^*_{n,z-t+1} - x^*_{n,z-t+2}\right) \\
&= \begin{cases} \delta^t + \alpha\left(2x^*_{n-k+1,0} - 1\right)\left(-\delta^{t-1}\right) & \text{if } t \text{ is even} \\ -\delta^t + \alpha\left(2x^*_{n-k+1,0} - 1\right)\delta^{t-1} & \text{if } t \text{ is odd} \end{cases} \\
&= \begin{cases} \delta^{t-1}\left[\delta - \alpha\left(2x^*_{n-k+1,0} - 1\right)\right] & \text{if } t \text{ is even} \\ -\delta^{t-1}\left[\delta - \alpha\left(2x^*_{n-k+1,0} - 1\right)\right] & \text{if } t \text{ is odd} \end{cases}
\end{aligned}$$

(4) For even z, we know that $\left(2x^*_{n-k+1,0} - 1\right) \geq 0$ because $x^*_{n,0} \geq 1/2$. Define $\Theta_k \equiv 2x^*_{n-k+1,0} - 1$ where $0 \leq \Theta_k \leq 1$. Note that $\alpha/\delta = e^{-r(\tau-\Delta)} \equiv \Xi$ where $0 \leq \Xi \leq 1$ if $\tau - \Delta \geq 0$ and $\Xi \geq 1$ if $\tau - \Delta \leq 0$. Hence, we have,

$$\begin{aligned}
x^*_{n-k,z-t} - x^*_{n-k,z-t+1} &= \begin{cases} \delta^{t-1}\left(\delta - \alpha\Theta_k\right) & \text{if } t \text{ is even} \\ -\delta^{t-1}\left(\delta - \alpha\Theta_k\right) & \text{if } t \text{ is odd} \end{cases} \\
&= \begin{cases} \delta^{t}\left(1 - \Xi\Theta_k\right) & \text{if } t \text{ is even} \\ -\delta^{t}\left(1 - \Xi\Theta_k\right) & \text{if } t \text{ is odd} \end{cases}
\end{aligned}$$

Hence, when $\tau - \Delta \geq 0$, we must have $x^*_{n-k,z-t} \geq x^*_{n-k,z-t+1}$ when t is even and $x^*_{n-k,z-t} \leq x^*_{n-k,z-t+1}$ when t is odd. If $\tau - \Delta \leq 0$, there exist a unique r^*_k such that when $r = r^*_k$, we have $x^*_{n-k,z-t} = x^*_{n-k,z-t+1}$;and that when $r < r^*_k$, $x^*_{n-k,z-t} \geq x^*_{n-k,z-t+1}$ when t is even and $x^*_{n-k,z-t} \leq x^*_{n-k,z-t+1}$ when t is odd; and the when $r > r^*_k$, $x^*_{n-k,z-t} \leq x^*_{n-k,z-t+1}$ when t is odd and $x^*_{n-k,z-t} \geq x^*_{n-k,z-t+1}$ when t is even.

(5) For odd z, there is a value $r^*_{\Theta_k}$ such that $r > r^*_{\Theta_k}$, then Θ_k is bounded by zero and one so that the analysis is similar to that in (4). However, if $r < r^*_{\Theta_k}$, then Θ_k is negative, and so that all analysis is reversed compared to (4). ■

Property (1) tells us that as long as stage and period are finite, if players are risk neutral and the set of agreement is a unit simplex, we must have a unique SPE. Property (2) shows that for even period model, as the future surplus drops, the equilibrium offers drops. It is because the enhancing effect of first-mover advantage and enhancing effect of last-mover advantage acting on the proposer. In such cases, accepting a proposal would earn an advantageous position in the next stage as a first proposer. As players value future surpluses, they are willing to make concession today strategically if first mover status is an advantageous position. Property (3) shows that first mover status is not always welcomed in model with odd period because enhancing effect of first-mover advantage and damping effect of last-mover advantage on the proposer. In such case, the second mover could have better standing if future surplus is large or players are patient. Therefore, under such situation, even the first mover may be willing to give up part of the interest, compared to one-stage bargaining, in order not to be in a disadvantageous position in the next stage. Property (4) and Property (5) tells us that the equilibrium offers in a stage would increase and decrease in alternating and damping manner. The direction for onset of swing would depend on the number of period in each stage z, difference of lags $\tau - \Delta$, number of stage remainning $n - k$ and the time preference r.

Strategic concession can cast an insight into models with incomplete information. In one-stage model, players without common knowledge of other player's preference might try to pretend himself as very patient player because the more patience the player has, the more surplus he can obtain in SPE. However, in multi-stage model, in some situation, more patient player would get less. Therefore, players will pretend themselves to be an *impatient* player so that other players could not extract his valuation over future surplus.

II Infinite stage bargaining $G(\infty, z)$

The result is almost the same as the infinite stage, infinite period model $G(\infty, \infty)$, so we will discuss the two models together in the next section.

5 Equilibrium for infinite period bargaining

We have shown that the requirement for uniqueness of perfect equilibrium in finite period game is not difficult to meet. However, for infinite period game, the set of SPE is not necessarily unique even if the Pareto frontier is continuous. We would like to adopt the security equilibrium concept proposed by Binmore[22] to solve for the set of perfect equilibrium outcome.

I Security equilibrium

Binmore's refinement procedure is closely related to the concept of iterated conditional dominance. The following definition is directly drawn from Fudenberg textbook [10].

> "(Iterated Conditional Dominance) In a multi-stage game with observed actions, action a_i^t is conditionally dominated at stage t given history h^t if,[25] in the subgame beginning at h^t, every strategy for player i that assigns positive probability to a_i^t is strictly dominated. Iterated conditional dominance is the process that, at each round, deletes every conditionally dominated action in every subgame, given the opponent's strategies that have survived the previous rounds."

It is obvious that no subgame-perfect strategy profile is removed by iterated conditional dominance in a finite- or infinite-horizon game of perfect information.[26] Therefore, the set of outcome that have survived from the iterated conditional dominance is no less than the set of SPE outcome. In general, two sets are not equal but Binmore [22], has established the

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[25] a_i^t means the action of player i available at stage t.

[26] The proof appears as exercise in textbook [10]. Obviously, for any subgame-perfect strategy profile, it cannot assign any positive probability to any strategy a_i^t which is strictly dominated; otherwise, players would benefit from the deviation to a strategy by assigning that probability to another strategy a_i^t which is not strictly dominated.

equality under a few conditions.[27] In Binmore's paper, he deal with surplus directly because players are risk neutral. This method still works in our characterization of bargaining game because payoff and strategy has bijective relationship.

As bargaining game is a sequential move game between two players, each period will have two subgames and hence we will do two eliminations in each period. Since the bargaining game could possibly end in any period, we need to do $2t$ rounds of elimination[28] for bargaining concluded at tth period where $t = 0, 1, 2, \ldots$. Denote $E_{0,t}$ be the result of elimination for the bargaining problem ended at time $t\Delta$ and $E_{s,r}$ (where $s + r = t$) to be the outcome survived the $2r$ rounds of elimination. Before any elimination, the set $E_{t,0}$ must be the feasible set at the beginning of the stage,[29] hence,

$$E_{t,0} = \mathcal{U}^{t\Delta} = \left\{ \left(u^{t\Delta}, v^{t\Delta} \right) : \left(U^{-1} \left(u^{t\Delta} / \delta_A^t \right), V^{-1} \left(v^{t\Delta} / \delta_B^t \right) \right) \in X \right\}.$$

With the above notation, we can redefine the concept of Iterated Conditional Dominance.

Definition 1 *(Iterated Conditional Dominance applied in bargaining)*[30] *Denote x-upper*

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[27] Binmore [22] has made the following assumptions in his paper.

1. The feasible set X_t is closed and bounded. Note that the feasible set is the set of utility pair and $X_t \subseteq \mathbb{R}^2$.
2. Pareto frontier Ω_t^e is connected. Note that the connectness forbids the Pareto frontier has any monotone increasing segment.
3. (shrinking cake) $X_s \cup X_t = X_s$ for all $s \geq t$.

[28] The two is due to elimination of two players.

[29] Note that the time for beginning of a stage is $T_{k-1} + \tau$ or $T_k - t_k$ but not T_k because T_k is the time for kth agreement reached.

[30] Note that this method is due to Binmore. We have only refined the procedure to suit our special needs.

bound, x-lower bound, y-upper bound and y-lower bound of the set $E^k_{s,r}$ to be:

$$\begin{aligned}
\sup\nolimits_X E_{s,r} &= \{x \in \mathbb{R} : \exists y \in \mathbb{R}, (x,y) \in E_{s,r} \text{ such that } x \geq a \text{ for all } (a,b) \in E_{s,r}\} \\
\inf\nolimits_X E_{s,r} &= \{x \in \mathbb{R} : \exists y \in \mathbb{R}, (x,y) \in E_{s,r} \text{ such that } x \leq a \text{ for all } (a,b) \in E_{s,r}\} \\
\sup\nolimits_Y E_{s,r} &= \{y \in \mathbb{R} : \exists x \in \mathbb{R}, (x,y) \in E_{s,r} \text{ such that } y \geq b \text{ for all } (a,b) \in E_{s,r}\} \\
\inf\nolimits_Y E_{s,r} &= \{y \in \mathbb{R} : \exists y \in \mathbb{R}, (x,y) \in E_{s,r} \text{ such that } y \leq b \text{ for all } (a,b) \in E_{s,r}\}.
\end{aligned}$$

Suppose the bargaining terminates at time $t\Delta$. Before any elimination, the set $E_{t,0} = \mathcal{U}^t$. Suppose t is even, it is player A to make a proposal (because player A makes offer at even period of time). Now we are doing elimination at $t-1$:

***1st elimination at** $t-1$**:** The maximum and minimum of player A could get at t are $\sup_X E_{t,0}$ and $\inf_X E_{t,0}$ respectively.*[31] *Therefore, moving one period backward, it is not optimal for player B to make any proposal outside this range; otherwise player A simply just waits to get higher payoff. Hence, we have $\sup_X E_{t-1,1} = \sup_X E_{t,0}$ and $\inf_X E_{t-1,1} = \inf_X E_{t,0}$.*

***2nd elimination at** $t-1$**:** In the light of this, when player B is making a proposal at time $t-1$, the maximum and minimum offer would then be $\Omega_B^{t-1}(\inf_X E_{t-1,1}) = \Omega_B^{t-1}(\inf_X E_{t,0})$ and $\Omega_B^{t-1}(\sup_X E_{t-1,1}) = \Omega_B^{t-1}(\sup_X E_{t,0})$ respectively where $v^t = \Omega_B^t(u)$ and $u = \Omega_A^t(v)$ are functions of Pareto frontier at time t.*

Finally, combined with the feasible set at time $t-1$, we have finished the two round of elimination. Now we are doing elimination at $t-2$:

***1st elimination at** $t-2$**:** The maximum and minimum of player B could get at t are $\sup_Y E_{t-1,1}$ and $\inf_Y E_{t-1,1}$ respectively. Therefore, moving one period backward, it is not optimal for player A to make any proposal outside this range; otherwise player A*

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[31] Note that all payoff are discounted to time 0 so that no further discounting action is needed.

simply just waits to get higher payoff. Hence, we have $\sup_Y E_{t-2,2} = \sup_Y E_{t-1,1}$ *and* $\inf_Y E_{t-2,2} = \inf_Y E_{t-1,1}$.

2nd elimination at $t-2$***:*** *In the light of this, when player* B *is making a proposal at time* $t-2$*, the maximum and minimum offer would then be* $\Omega_A^t(\inf_Y E_{t-2,2}) = \Omega_A^t(\inf_Y E_{t-,0})$ *and* $\Omega_A^t(\sup_Y E_{t-2,2}) = \Omega_A^t(\sup_Y E_{t-2,2})$ *respectively.*

The process repeated in the same manner until $t=0$*. For* t *to be odd, the process of elimination at* $t=0$ *is same as the process of* $t=0$ *when* t *is odd. However, the process at* $t-2k$ *when* t *is odd would be the same as the process of* $t-(2k\pm1)$ *when* t *is even , and the the process at* $t-(2k\pm1)$ *when* t *is odd would be the same as the process of* $t-2k$ *when* t *is even. Hence,* $E_{s,r}$ *is inductively defined as when* $(t+r) \bmod 2 = 1$*, then*

$$E_{s,r} = \left\{ \begin{array}{c} (x,y)\in\mathbb{R}^2 : \inf_X E_{s+1,r-1} \le x \le \sup_X E_{s+1,r-1} \\ \text{and } \Omega_B^{t-r}(\sup_X E_{s+1,r-1}) \le y \le \Omega_B^{t-r}(\inf_X E_{s+1,r-1}) \end{array} \right\} \cap \mathcal{U}^{t-s}$$

and when $(t+r) \bmod 2 = 0$

$$E_{s,r} = \left\{ \begin{array}{c} (x,y)\in\mathbb{R}^2 : \Omega_A^{t-r}(\sup_Y E_{s+1,r-1}) \le x \le \Omega_A^{t-r}(\inf_Y E_{s+1,r-1}) \\ \text{and } \inf_Y E_{s+1,r-1} \le y \le \sup_Y E_{s+1,r-1} \end{array} \right\} \cap \mathcal{U}^{t-s}.$$

To complete the procedure, we need to perform the elimination for all outcomes ended in all time form t $=0,1,2.....$ *and so that the set of outcome survived iterated conditional dominance is*

$$E = \bigcap_{t=0}^{\infty} E_{0,t}.$$

Remark 3 *If we assume* $\{X_k \cup \{d_k\}\}$ *remains constant in the kth stage and if discount factor* r_A *and* r_B *remain unchanged, then* $E_{0,t_1} \subseteq E_{0,t_2}$ *for any* $t_1 \ge t_2$*. Hence, the necessary and sufficient condition for* E *being singleton is that* $E_{0,\infty}$ *is a singleton.*

II Finite stage bargaining $G(n,\infty)$

For one-stage bargaining $G(1,\infty)$, the basic case has been analyzed by Rubinstein in his classical paper [5]. The non-convex feasible case has been analyzed by Binomore and Herro [23] using the concept of security equilibrium, which we have shown in the previous subsection. Now for multi-stage bargaining $G(n,\infty)$, we wish to solve it using the principle of backward induction. Given the result of final stage, we solve for the second to the last stage and so on. If every stage could be solved uniquely, we should expect there will be a unique SPE for the whole game. Indeed, this result is true but needs extra conditions.

Proposition 3 *For bargaining problem $G(n,\infty)$, if each bargaining stage itself has a unique SPE with immediate agreement and satisfy certain regularity conditions(see the proof), the whole game has a unique SPE with immediate agreements and the supporting strategy is stage-stationary.*

Proof. To facilitate the discussion, we only deal with two-stage cases. An induction argument would complete the proof for finitely many stages cases. Suppose the last stage equilibrium is (a,b) if the first proposer is player A and is (c,d) if the first proposer is player B.[32] Since (a,b) and (c,d) are equilibrium solution, they must be on the Pareto frontier. For immediate agreement, we must have $\delta_A a \geq c$ and $\delta_B d \geq b$. Otherwise, player would choose to wait. Further suppose the unique strategy is stationary, then we have $c \geq \delta_A a$ and $b \geq \delta_B d$. Hence, we must have $\delta_A a = c$ and $\delta_B d = b$. Now consider the first stage. First, as we have found out the last stage solution, we could simply combine the outcome from the second stage to the first stage to form a new game because that second

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[32] In general, first mover should have absolute advantage in infinite period game. Hence, though we have not adopted the assumption, we usually assume $a \geq b$ and $c \leq d$.

stage result is unique. Thus, the new(cumulative) Pareto Frontiers would be:

$$\begin{aligned}\Omega_1^{\tilde{T}_1} &= \Omega_1^{T_1} + \Omega_2^{T_2} \\ &= \begin{cases} \Omega_1^{t_1} + \left(\alpha_A \delta_A^{t_1/\Delta} a, \alpha_B \delta_B^{t_1/\Delta} b\right) & \text{if } T_1 = 2k\Delta \\ \Omega_1^{t_1} + \left(\alpha_A \delta_A^{t_1/\Delta} c, \alpha_B \delta_B^{t_1/\Delta} d\right) & \text{if } T_1 = (2k+1)\,\Delta \end{cases}\end{aligned}$$

because $t_1 = T_1$ and the new(cumulative) feasible set $\mathcal{W}$ at time t_1:

$$\mathcal{W}^{t_1} = \begin{cases} \mathcal{U}_1^{t_1} + \left(\alpha_A \delta_A^{t_1/\Delta} a, \alpha_B \delta_B^{t_1/\Delta} b\right) & \text{if } T_1 = 2k\Delta \\ \mathcal{U}_1^{t_1} + \left(\alpha_A \delta_A^{t_1/\Delta} c, \alpha_B \delta_B^{t_1/\Delta} d\right) & \text{if } T_1 = (2k+1)\,\Delta \end{cases}.$$

Now, the two-stage bargaining problem becomes a one-stage bargaining problem. Hence, we now could solve the two-stage bargaining problem using the concept of security equilibrium. For the new game, if we want uniqueness, we need the set E to be singleton. This is equivalent to require that (A) the set $E_{0,t}$ is shrinking as t increases, that is $E_{0,t_1} \subseteq E_{0,t_2}$ for $t_1 \geq t_2$ and (B) $E_{0,t}$ to be singleton as $t \to \infty$.

Intuitively, from geometric point of view, as the shape does not change, elimination procedure should return a unique point for the set E in the combined bargaining problem. Since the new frontiers are formed by transpose of original frontier, it is clear that the shape of the curve should be the same. From the pattern of the shift, we know that that the frontiers of $t = 2(k+1)\Delta$ move relatively upwards and frontiers of $t = 2k\Delta$ move relatively rightwards. As the shape does not change, elimination procedure should return a unique point for the set E in the combined bargaining problem. We would shows the step to arrive the solution formally.

(A): First, let $\xi_{t_1} = \sup_X \Omega_1^{t_1}$ be the maximum utility obtained by player A at time t_1, $\lambda_{t_1} = \sup_Y \Omega_1^{t_1}$ be the maximum utility obtained by player B at time t_1, $\theta_{t_1} = \inf_X \Omega_1^{t_1}$ be the minimum utility obtained by player A at time t_1 and $\beta_{t_1} = \inf_Y \Omega_1^{t_1}$ be the minimum utility obtained by player B at time t_1. By the assumption of underlying conflict of interest between players, $\left(\xi_{t_1}, \beta_{t_1}\right)$ and $(\theta_{t_1}, \lambda_{t_1})$ should lie on the $\Omega_1^{t_1}$:

$$\left(\sup_X \Omega_1^{\tilde{T}_1}, \inf_Y \Omega_1^{\tilde{T}_1}\right) = \begin{cases} \left(\xi_{t_1} + \alpha_A \delta_A^{t_1/\Delta} a, \beta_{t_1} + \alpha_B \delta_B^{t_1/\Delta} b\right) & \text{if } t_1 = (2k+1)\,\Delta \\ \left(\xi_{t_1} + \alpha_A \delta_A^{t_1/\Delta} c, \beta_{t_1} + \alpha_B \delta_B^{t_1/\Delta} d\right) & \text{if } t_1 = 2k\Delta \end{cases}$$

and

$$\left(\inf_X \Omega_1^{\tilde{T}_1}, \sup_Y \Omega_1^{\tilde{T}_1}\right) = \begin{cases} \left(\theta_{t_1} + \alpha_A \delta_A^{t_1/\Delta} a, \lambda_{t_1} + \alpha_B \delta_B^{t_1/\Delta} b\right) & \text{if } t_1 = (2k+1)\,\Delta \\ \left(\theta_{t_1} + \alpha_A \delta_A^{t_1/\Delta} c, \lambda_{t_1} + \alpha_B \delta_B^{t_1/\Delta} d\right) & \text{if } t_1 = 2k\Delta \end{cases}.$$

For immediate agreement, we need to have

$$\inf_Y \Omega_1^{\tilde{T}_1'} \leq \inf_Y \Omega_1^{\tilde{T}_1} \text{ and } \sup_Y \Omega_1^{\tilde{T}_1} \leq \sup_Y \Omega_1^{\tilde{T}_1'} \text{ where } T_1 = (2k+1)\,\Delta,\ T_1' = T_1 - \Delta$$

and

$$\inf_X \Omega_1^{\tilde{T}_1'} \leq \inf_X \Omega_1^{\tilde{T}_1} \text{and } \sup_X \Omega_1^{\tilde{T}_1} \leq \sup_X \Omega_1^{\tilde{T}_1'} \text{ where } T_1 = 2k\Delta,\ T_1' = T_1 - \Delta.$$

Hence, we need to have, when $t_1' = t_1 - \Delta$ and $t_1 = (2k+1)\,\Delta$,

$$\beta_{t_1'} + \alpha_B \delta_B^{t_1'/\Delta} b \leq \beta_{t_1} + \alpha_B \delta_B^{t_1/\Delta} d \text{ and } \lambda_{t_1} + \alpha_B \delta_B^{t_1/\Delta} d \leq \lambda_{t_1'} + \alpha_B \delta_B^{t_1/\Delta} b$$

and when $t_1 = t_1' - \Delta$ and $t_1 = 2k\Delta$

$$\theta_{t_1'} + \alpha_A \delta_A^{t_1'/\Delta} a \leq \theta_{t_1} + \alpha_A \delta_A^{t_1/\Delta} c \text{ and } \xi_{t_1'} + \alpha_A \delta_A^{t_1'/\Delta} c \leq \xi_{t_1} + \alpha_A \delta_A^{t_1/\Delta} a.$$

It can be easily shown that the above inequalities hold when

$$\begin{aligned}\beta_0\left(\delta_B-1\right)+\alpha_A\left(\delta_B d-b\right) &\geq 0,\\ \lambda_0\left(1-\delta_B\right)+\alpha_A\left(b-\delta_B d\right) &\geq 0,\\ \theta_0\left(\delta_A-1\right)+\alpha_B\left(\delta_A c-a\right) &\geq 0,\\ \alpha_0\left(1-\delta_A\right)+\alpha_B\left(a-\delta_A c\right) &\geq 0.\end{aligned}$$

These conditions are easily satisfied when players are patient enough and the unique equilibrium is stationary. Usually we have $\beta_0=\theta_0=0$ as we set disagreement payoff equal to perpetual disagreement payoff which is zero. Since the shape of $\Omega_1^{\tilde{T}_1}$ are the same for every value of $\tilde{T}_1$,there would be no intersection point between Pareto frontiers $\Omega_1^{\tilde{T}_1}$ across time. Hence, the size of the set $E_{0,t}$ is non-increasing as t increases.

(B):To prove this formally, let us look at the difference between the new Pareto frontiers $\Omega_1^{\tilde{T}_1}$ and the first-stage Pareto frontiers $\Omega_1^{T_1}$. Since in the original problem has unique equilibrium, it means that the curves must be in the shape such that the set E contains only one element. Given $\{X_1\cup\{d_1\}\}$ remains constant throughout the whole game, from the above remark, we also know that $E_{0,\infty}$ is also a singleton. For simplicity, let us assume common discount factor $r_A=r_B=r$ such that $\delta_A=\delta_B=\delta$ and maximum surplus for each player would be one. Now, if we look at new Pareto frontiers $\Omega_1^{\tilde{T}_1}$, we could see that new Pareto frontier functions are:

New Frontier Function	t is odd	t is even
$u=\Omega_A^{\tilde{T}_1}(v)$	$\Omega_A^{T_1}\left(v-\alpha\delta^{T_1}c\right)+\alpha\delta^{T_1}d$	$\Omega_A^{T_1}\left(v-\alpha\delta^{T_1}a\right)+\alpha\delta^{T_1}b$
$v=\Omega_B^{\tilde{T}_1}(v)$	$\Omega_B^{T_1}\left(v-\alpha\delta^{T_1}d\right)+\alpha\delta^{T_1}c$	$\Omega_B^{T_1}\left(v-\alpha\delta^{T_1}b\right)+\alpha\delta^{T_1}a$

When t is odd, we would have

$$\sup\nolimits_X E_{0,t}=\Omega_A^{\tilde{0}}\left(...\Omega_A^{\widetilde{t-3}}\left(\Omega_B^{\widetilde{t-2}}\left(\Omega_A^{\widetilde{t-1}}\left(\alpha\delta^t d\right)\right)\right)\right)=\Gamma\left(\delta^t d\right)$$

and

$$\inf\nolimits_X E_{0,t} = \Omega_A^{\tilde{0}} \left(...\Omega_B^{\widetilde{t-3}} \left(\Omega_A^{\widetilde{t-2}} \left(\Omega_B^{\widetilde{t-1}} \left(\delta^t \left(1+d\right)\right)\right)\right)\right) = \Gamma\left(\delta^t \left(1+\alpha d\right)\right)$$

where $\Gamma\left(\cdot\right) = \Omega_A^{\tilde{0}} \left(...\Omega_A^{\widetilde{t-3}} \left(\Omega_B^{\widetilde{t-2}} \left(\Omega_A^{\widetilde{t-1}} \left(\cdot\right)\right)\right)\right)$ is a function of t. Assume that $\Gamma\left(\cdot\right)$ is twice continuous differentiable, we can approximate the difference using second order of Taylor series expansion directly:

$$\begin{aligned} & \Gamma\left(\delta^t \left(1+\alpha d\right)\right) - \Gamma\left(\delta^t \alpha d\right) \\ = \; & \Gamma'\left(\delta^t \alpha d\right)\left(1+\alpha d\right) \triangle t \delta^{2t} + \frac{\Gamma''\left(\delta^t \alpha d\right)}{2!}\left[\left(1+\alpha d\right)\triangle t\right]^2 \delta^{3t} + \ldots \\ = \; & 0 \text{ when } t \to \infty \end{aligned}$$

because $\Gamma'\left(\delta^t \alpha d\right), \Gamma''\left(\delta^t \alpha d\right), \ldots$ should remains finite as slope of frontier cannot be infinite.

Now, consider when t is even,

$$\sup\nolimits_X E_{0,t} = \Omega_A^{\tilde{0}} \left(...\Omega_B^{\widetilde{t-3}} \left(\Omega_A^{\widetilde{t-2}} \left(\Omega_B^{\widetilde{t-1}} \left(\alpha \delta^t a \right)\right)\right)\right) = \Lambda\left(\delta^t \alpha a\right)$$

and

$$\sup\nolimits_X E_{0,t} = \Omega_A^{\tilde{0}} \left(...\Omega_B^{\widetilde{t-3}} \left(\Omega_A^{\widetilde{t-2}} \left(\Omega_B^{\widetilde{t-1}} \left(\delta^t \left(1+\alpha a\right)\right)\right)\right)\right) = \Lambda\left(\delta^t \left(1+\alpha a\right)\right).$$

Using Taylor series expansion, we find that

$$\begin{aligned} & \Lambda\left(\delta^t \left(1+\alpha a\right)\right) - \Lambda\left(\delta^t \alpha a\right) \\ = \; & \Lambda'\left(\delta^t \alpha a\right)\left(1+d\right) \triangle t \delta^{2t} + \frac{\Lambda''\left(\delta^{t\alpha} a\right)}{2!}\left[\left(1+d\right)\triangle t\right]^2 \delta^{3t} + \ldots \\ = \; & 0 \text{ when } t \to \infty \end{aligned}$$

because $\Lambda'\left(\delta^t \alpha a\right), \Lambda''\left(\delta^t \alpha a\right), \ldots$should remains finite as slope of frontier cannot be infinite. Now, we have shown that the even t set of $E_{0,t}$ and odd set of $E_{0,t}$ converge to a point what remains to be shown is that two points coincide in the limit.

$$\Lambda\left(\delta^t \alpha a\right) = \Gamma\left(\delta^t \alpha d\right) \text{ as } t \to \infty.$$

Suppose there is symmetry between two players, then we must have $a = d$, $b = c$ and $\Lambda = \Gamma$ for all t.[33] Then it would be true that $\sup_X E_{0,t} = \inf_X E_{0,t}$ as $t \to \infty$ and $\sup_Y E_{0,t} = \inf_Y E_{0,t}$ as $t \to \infty$ can be shown using similar logic.

From the construction of the strategy, it is clear that strategy would be same for each stage because the environment has not changed across period so that the strategy is stage-period stationary. ■

Different from the finite-stage, finite-period model, last-mover advantage does not exist in infinite period model because there is no last mover so there is no ultimatum. Therefore, only first-mover advantage exists under all circumstances. Similar to result from the finite counterpart, the effect due to first-mover advantage would be greater in the first few stages and this advantage would be reduced as there are less number of stages in the future. The intuition of this is similar to the finite case that the more future cooperation possibility is, the both players are willing to sacrifice the present surplus in order not to delay the future total surplus. Going back to the standard Rubinstein bargaining problem could allow us to observe this fact easily. When we looking at the pattern of equilibrium offer, one could observe the equilibrium offers are decreasing as the game proceeds. See appendix II for the proof using Shaked and Sutton's technique. One can easily show that the result of the finite period model is equal to the limiting case of infinite period model. Figure 1 shows how the equilibrium offers change as number of remaining stages for finitely repeated Rubinstein bargaining game. Note that since the strategy are stage-stationary, the strategy for each

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[33] Note that it is only sufficient conditions. In reality, the necessary condition required is that the two sets have to converge in the limit.

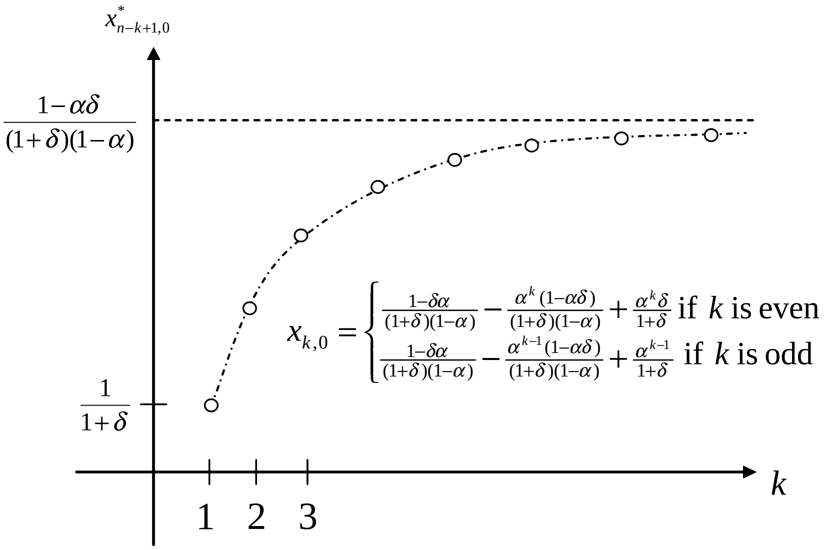

Figure 1: Stage-stationary equilibrium offer as a function of the number of remaining stage

player in each stage is the same.

The reason for the decreasing offers is that the first proposer could extract lesser surplus as cost due to delay is dropping. Hence, the ehhancing effect of first-mover advantage drops when there is fewer number of stage remaining. Before ending this section, we would like to make some comments on the conclusion by Muthoo's conclusion [2]:

> "...Rubinstein's unique equilibrium is not robust to "small external effects", in that the indeterminacy of the basic bargaining problem is re-obtained if the players expect to bargain, with an arbitrarily small probability, over the partition of another cake each time they reach agreement over the partition of an exisiting cake."

From the result obtained in the section, we might append the following statement: "*However, when there is a limit over maximum number of cakes can be divided or both players know in advance they have a finite number of cakes to divide, the uniqueness is* *re-obtained."*

III Infinite stage bargaining $G(\infty,\infty)$

We will establish two results in this subsection: (i) refine the set of SPE by removing extremal outcomes using global stability criteria and (ii) support outcomes in convexified set which may be outside the feasible set.

From repeated game literature, using folk theorem, we could establish that any payoff which is strictly greater than the minimax payoff to be a perfect equilibrium for any infinitely repeated game as long as players are patient enough. In infinite-stage, infinite-period models $G(\infty,\infty)$ (and also $G(\infty,z)$ [34]), following the same principle, one could show that any result strictly greater than the disagreement payoff could be supported as SPE. Moreover, due to the structure of a bargaining game, outcomes with payoff equal to minimax level could also be supported.[35] However, it means that extremal outcome such as one player always grab the whole surplus can be supported, which is rather counter-intuitive.[36] As the set of SPE is so huge that almost all possible outcomes could be supported, we need another stricter and natural criteria to remove those outcomes. Since infinitely repeated game usually involves infinitely many strategies with long-period histories and complicated punishment scheme, selection of SPE would be more natural and practical for implementation if we have simple strategies, simple punishment scheme, limited dependence on history and robustness of equilibrium payoff against deviation. The first three requirements could be satisfied if we adopt the penal code for punishment mechanism [9]. The notion of global stability by Kandori [25] could serve the purpose of removing outcomes which would lead to off-equilibrium payoff forever.[37] Using penal code and global stability, we could eliminate those extremal outcomes. The intuition is that one could not find suitable punishment be-

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[34] Since the proof is very similar, we would omit it in exposition.

[35] Muthoo [2] has proved this result for infinitely repeated Rubinstein bargaining game.

[36] The reason is that if one player is being exploited or bullied by another player, he should not cooperate with him or should seek another to share the surplus.

[37] Roughly speaking, global stability means that off-equilibrium path can not last forever in the sense that no matter under what kind of history, players must eventually receive the equilibrium payoff.

cause the outcome is already worst for one player: "How can one possibly use punishment to punish a player obtaining the lowest possible payoff?"

Proposition 4 *For bargaining problem* $G(\infty, \infty)$ *with* $\{X_k \cup \{d_k\}\}_{k=1}^{n}$ *, if it satisfies the assumption of underlying conflict of interest, every outcome could be supported as SPE if players are patient enough or time lags between offers and time lags between stages are short enough.*[38] *If we refine the set of equilibrium using global stability, extremal outcome that one player always obtains his maximum possible surplus and another player always his minimal possible surplus in all stages could be excluded.*

Proof. To prove the proposition, we first need to establish that every possible outcome can be supported as perfect equilibrium using penal code and then exclude the extremal outcome using stability criteria. Now, with the assumption of underlying conflict of interest, let $\dot{\mathrm{u}}_k$ and $\underset{\cdot}{\mathrm{u}}_k$ player A's highest share of surplus and lowest share of surplus obtained in kth stage respectively. Similarly, $\dot{\mathrm{v}}_k$ and $\underset{\cdot}{\mathrm{v}}_k$ player B's highest share of surplus and lowest share of surplus obtained in kth stage respectively. Under the assumption as underlying conflict of interest, when player A is having $\dot{\mathrm{u}}_k$, player B is having $\underset{\cdot}{\mathrm{v}}_k$ and when player A is having $\underset{\cdot}{\mathrm{u}}_k$, player B is having $\dot{\mathrm{v}}_k$. To support outcome with immediate agreements $\langle (x_1, y_1), T_1, (x_2, y_2), T_2, ... \rangle$ where $T_k = (k-1)\tau$, it simply requires that in kth stage, player A only accepts offers not less than x_k and rejects any other offers, and player B only accepts offers not less than y_k and rejects any other offers. Punishment scheme would start immediately if any player accept any offers less than the prescribed amount.

We would like to use penal code [9] to formulate the punishment scheme. If any player deviates, he would receive penalty designed for him. If the deviant fails to conform the penal code, the punishment would restart from the beginning of the penal code. If any player fails to punish the deviants, that player would be punished by the penalty designed

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[38] Muthoo [2] only proves this fact in standard Rubinstein game but not general setting.

for him. Note that when both players deviates at the same time, there is no punishment needed. Roughly speaking, our penalty is that the deviant would receive lowest possible payoff and the punisher would receive highest payoff so that no player has incentive to deviate and player do have incentive to conform to punishment. Formally, the penal code for player A is that player B offers $\underset{\cdot}{\mathrm{u}}_k$ and retains $\dot{\mathrm{v}}_k$ and player A has to conform this division scheme from now and on. Similarly, the penal code for player B is that player A always offers $\underset{\cdot}{\mathrm{y}}_k$ and retains $\dot{\mathrm{u}}_k$ and player B has to conform this division scheme from now and on.

On the equilibrium path, given players equilibrium strategy, it would not be optimal for players to offer anything other than the prescribed. When one player deviates to offer more, the other player is willing to accept such offers and the game would proceeds as in no deviation. When one player deviates to offer less, the other player would reject if the gain is less than the loss from perpetual punishment of lowest possible payoff:

$$\begin{aligned}
\delta_A \sum_{h=k}^{\infty} U\left(x_h\right) \alpha_A^h &\geq U\left(x_k\right)+\sum_{h=k+1}^{\infty} U\left(\underset{\cdot}{\mathrm{u}}_h\right) \alpha_A^h \text{ and} \\
\delta_B \sum_{h=k}^{\infty} V\left(y_h\right) \alpha_B^h &\geq V\left(y_k\right)+\sum_{h=k+1}^{\infty} V\left(\underset{\cdot}{\mathrm{y}}_h\right) \alpha_B^h .
\end{aligned}$$

If the conditions are not met, the game would proceeds to the execution of penal code.

When the game is under the penal code of player A, as player A is receiving the lowest possible payoff, it makes no sense for player B to offer more as player A would simply accept it. Player A makes an offer less than $\dot{\mathrm{v}}_k$, player B would reject when the one-shot gain is less than the long-run loss if

$$\delta_B \sum_{h=k}^{\infty} V\left(\dot{\mathrm{v}}_h\right) \alpha_B^h \geq V\left(\dot{\mathrm{v}}_k\right)+\sum_{h=k+1}^{\infty} V\left(\underset{\cdot}{\mathrm{y}}_h\right) \alpha_B^h .$$

Of course, as player A is getting the minimal and player B is getting the maximal, player

A cannot offer more and player B cannot offer less. The deduction for game under penal code of player B goes similarly. The sufficient condition required for no deviation would be

$$\delta_A \sum_{h=k}^{\infty} U\left(\dot{\mathrm{u}}_h\right)\alpha_A^h \geq U\left(\dot{\mathrm{u}}_k\right) + \sum_{h=k+1}^{\infty} U\left(\underset{\cdot}{\mathrm{u}}_h\right)\alpha.$$

If $x_k = x$, $y_k = y$, $\dot{\mathrm{v}}_k = \dot{\mathrm{v}}$ and $\dot{\mathrm{u}}_k = \dot{\mathrm{u}}$ for all k, then sufficient conditions would become

$$\begin{array}{rcl}
\left(\delta_A + \alpha_A - 1\right) U\left(x\right) - \alpha_A U\left(\underset{\cdot}{\mathrm{u}}_{k+1}\right) & \geq & 0, \\
\left(\delta_B + \alpha_B - 1\right) V\left(y\right) - \alpha_B V\left(\underset{\cdot}{\mathrm{v}}_{k+1}\right) & \geq & 0, \\
\left(\delta_B + \alpha_B - 1\right) V\left(\dot{\mathrm{v}}\right) - \alpha_B V\left(\underset{\cdot}{\mathrm{v}}_{k+1}\right) & \geq & 0, \\
\left(\delta_A + \alpha_A - 1\right) U\left(\dot{\mathrm{u}}\right) - \alpha_A U\left(\underset{\cdot}{\mathrm{u}}_{k+1}\right) & \geq & 0.
\end{array}$$

Usually, we would have $V\left(\underset{\cdot}{\mathrm{v}}_k\right) = 0$ and $U\left(\underset{\cdot}{\mathrm{u}}_k\right) = 0$ for all k, then we would return to the sufficient conditon specified by Muthoo:

$$\delta_A + \alpha_A \geq 1 \text{ and } \delta_B + \alpha_B \geq 1.$$

If we want to support outcomes without immediate agreement,[39] we only need to make players offering lowest possible share to the other player and only accepts highest possible share from other player before the prescribed time and return the prescribed division scheme after the prescribe time. On the equilibrium path, if anyone deviates at time T_k' (or at t_k' if the time is counted from beginning of the stage) and accepts, he would be

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[39] That is, outcome $\left\langle \left(x_1, y_1\right), T_1, \left(x_2, y_2\right), T_2, \ldots \right\rangle$ with $T_k \neq \left(k-1\right)\tau$.

punished forever and so the conditions required are:

$$\begin{aligned}
\sum_{h=k}^{\infty} U\left(x_h\right)\delta_A^{t_h/\Delta}\alpha_A^h &\geq \delta_A^{t'_k/\Delta}U\left(\dot{\mathrm{u}}_k\right)+\delta_A^{t'_k/\Delta}\sum_{h=k+1}^{\infty}U\left(\underset{\cdot}{\mathrm{u}}_h\right)\delta_A^{t_h/\Delta}\alpha_A^h \text{ and}\\
\sum_{h=k}^{\infty} V\left(y_h\right)\delta_B^{t_h/\Delta}\alpha_B^h &\geq \delta_B^{t'_k/\Delta}V\left(\dot{\mathrm{v}}_k\right)+\delta_B^{t'_k/\Delta}\sum_{h=k+1}^{\infty}V\left(\underset{\cdot}{\mathrm{v}}_h\right)\delta_B^{t_h/\Delta}\alpha_B^h.
\end{aligned}$$

Again, when $t_k = t\Delta$, $x_k = x$, $y_k = y$, $\dot{\mathrm{v}}_k =\dot{\mathrm{v}}$, $\dot{\mathrm{u}}_k =\dot{\mathrm{u}}$, $V\left(\underset{\cdot}{\mathrm{v}}_k\right) = 0$ and $U\left(\underset{\cdot}{\mathrm{u}}_k\right) = 0$ for all k, we have

$$\begin{aligned}
\delta_A^t U\left(x\right)-\left(1-\alpha\right)U\left(\dot{\mathrm{u}}\right) &\geq 0 \text{ and}\\
\delta_B^t V\left(y\right)-\left(1-\alpha\right)V\left(\dot{\mathrm{v}}\right) &\geq 0
\end{aligned}$$

which are similar to above conditions but more difficult to satisfied. Both set of inequalities would be same only if equalities holds for $U\left(x\right) \leq U\left(\dot{\mathrm{u}}\right)$ and $V\left(y\right) \leq V\left(\dot{\mathrm{v}}\right)$.

We have proved the first part of the statement that every outcome could be supported and now we have to adopt the globally stable criteria. Since the stability requires payoff returns to equilibrium payoff eventually, therefore punishment cannot last forever and must end within finite number stages. Let p be the number of stage of punishment. Using the same logic, we could find out that the inequalities to be satisfied are:

$$\begin{aligned}
\delta_A\sum_{h=k}^{\infty} U\left(x_h\right)\alpha_A^h &\geq U\left(x_k\right)+\sum_{h=k+1}^{k+p}U\left(\underset{\cdot}{\mathrm{u}}_h\right)\alpha_A^h+\sum_{h=k+p+1}^{\infty}U\left(x_h\right)\alpha_A^h \text{ and}\\
\delta_B\sum_{h=k}^{\infty} V\left(y_h\right)\alpha_B^h &\geq V\left(y_k\right)+\sum_{h=k+1}^{k+p}V\left(\underset{\cdot}{\mathrm{v}}_h\right)\alpha_B^h+\sum_{h=k+p+1}^{\infty}V\left(y_h\right)\alpha_B^h.
\end{aligned}$$

And when it is on off-equilibrium path, for the deviation made at time $t''_k\Delta$ ($t''_k \leq p$, counted from the starting of penal code), the conditions are

$$\begin{aligned}
\delta_A \sum_{t''_k}^{p} U\left(\dot{\mathrm{u}}_h\right) \alpha_A^h + \delta_A \sum_{p+1}^{\infty} U\left(x_h\right) \alpha_A^h &\geq \alpha_A^{t''_k} U\left(\dot{\mathrm{u}}_k\right) + \sum_{t''_k+1}^{t''_k+p} U\left(\underset{\cdot}{\mathrm{u}}_h\right) \alpha_A^h + \sum_{t''_k+p+1}^{\infty} U\left(x_h\right) \alpha_A^h, \\
\delta_B \sum_{t''_k}^{p} V\left(\dot{\mathrm{v}}_h\right) \alpha_B^h + \delta_B \sum_{p+1}^{\infty} V\left(x_h\right) \alpha_B^h &\geq \alpha_A^{t''_k} V\left(\dot{\mathrm{v}}_k\right) + \sum_{t''_k+1}^{t''_k+p} V\left(\underset{\cdot}{\mathrm{v}}_h\right) \alpha_B^h + \sum_{t''_k+p+1}^{\infty} V\left(x_h\right) \alpha_B^h.
\end{aligned}$$

Now, let us simplify the inequalities by, $x_k = x$, $y_k = y$, $\dot{\mathrm{v}}_k =\dot{\mathrm{v}}$, $\dot{\mathrm{u}}_k =\dot{\mathrm{u}}$, $V\left(\underset{\cdot}{\mathrm{v}}_k\right) = 0$ and $U\left(\underset{\cdot}{\mathrm{u}}_k\right) = 0$ for all k, we have

$$\begin{aligned}
\delta_A + \alpha_A - \alpha_A^p &\geq 1, \\
\delta_B + \alpha_B - \alpha_B^p &\geq 1, \\
\alpha_A \delta_A \left(1 - \alpha_A^p\right) x - \dot{\mathrm{u}} \left(1 - \alpha_A\right) \left(1 - \delta_A\right) &\geq 0, \\
\alpha_B \delta_B \left(1 - \alpha_B^p\right) y - \dot{\mathrm{v}} \left(1 - \alpha_B\right) \left(1 - \delta_B\right) &\geq 0.
\end{aligned}$$

Clearly if $x = 0$ or $y = 0$, the last two equation can never be satisfied no matter how large α and δ are. Intuitively speaking, it is because punishment has no deterrent effect on the player who always accepts lowest possible share. Then that player would defect if he is given an offer with non-zero payoff. The reduction of set of perfect equilibrium is surprisingly analog to the standard result from folk theorem that the payoff vector has no component equal to the component of minimax payoff vector. ■

Therefore, using the concept of global stability, we have shown that extremal outcomes are removed from the set of perfect equilibrium. Note that the first two inequalities tell us that if we are to use limited stage punishment scheme, the patience and/or the time lag requirement are/is more stringent. Of course when the number of punishment stage goes to infinity, the condition converges to the Muthoo's requirement $\alpha + \delta \geq 1$. Moreover, the last two equations show that limited punishment also links the size of the set of equilibrium to equilibrium partition level, in addition to our usual requirements of patience and time lag.

Now, we are going to show the second important result that if the bargaining set is non-convex, outcome outside the feasible set could be support non-stationary SPE outcome. Actually, this is the corollary of the proposition 4.

Corollary 5 *For bargaining problem* $G(\infty, n)$ *or* $G(\infty, \infty)$*, if the cumulative feasible sets* $\mathcal{U}_n^{\tilde{T}_n} = \left(\sum_{k=1}^{n} u_k^{T_k}, \sum_{k=1}^{n} v_k^{T_k}\right)$ *where* $\left(u_k^{T_k}, v_k^{T_k}\right) \in \mathcal{U}_k^{T_k}$ *are non-convex, then any outcome in the convexified set can be achieved as close as possible if players are patient enough or time lag between offers and stage are short enough.*

Proof. From the proposition that every outcome could be supported as perfect equilibrium if players are patient enough or time lag between offers and stage are short enough. For outcome with payoff (u, v) where $u = ta + (1-t)\, c$ and $v = tb + (1-t)\, d$, $0 \leq t \leq 1$,$(a, b) \in \mathcal{U}_k^{\tilde{T}_k}$ and $(c, d) \in \mathcal{U}_k^{\tilde{T}_k}$ to be supported by playing (a, b) for $n_{a,b}$ times and then play (c, d) for $n_{c,d}$ times, we need to have

$$\frac{t}{1-t} = \frac{1-\alpha_A^h}{\alpha_A^h\left(1-\alpha_A^k\right)} = \frac{1-\alpha_B^h}{\alpha_B^h\left(1-\alpha_B^k\right)}$$

where $n_{a,b} = h/\left(h+k\right)$ and $n_{c,d} = k/\left(h+k\right)$. Since $n_{a,b}$ and $n_{c,d}$ are required to be integers but h and k are real numbers between zero and one, we could achieve the particular outcome exactly only when h and k are rational number, though we could approach as close as possible if h or k is real number[40]. ■

Besides the refinement by stability applied above, one common criterion is efficiency consideration. For repeated bargaining problems, efficiency could be viewed at least from two aspects: static efficiency and dynamic efficiency[41]. Static efficient solution is that one

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[40] It is because for any two unequal rational number, there is a real number in between.

[41] Dynamic efficiency has been mentioned by [3] on infinitely repeated game.

cannot improve payoff of one stage game of one player without harming the other player. Clearly, under the assumption of underlying conflict of interest, any agreement on Pareto frontier are static efficient. On the other hand, dynamic efficient solution is that one cannot increase the sum of discounted payoffs of the whole game of one player without harming the other player. In the other words, static efficiency is Pareto efficiency within a stage while dynamic efficiency is Pareto efficiency for the whole game.[42] Hence, one dynamic concept does not imply the other.

In one-stage bargaining, in almost all cases, static efficiency could be assumed to be true.[43] In multi-stage bargaining problem, many static efficient solutions are not dynamic inefficient. Particularly, for non-convex set, outcomes outside the feasible set are more dynamic efficient than all solutions lie on Pareto frontier. We all know that efficiency does not directly imply absence of implementation problems in practice. From the corollary, it is obvious that at least two different offers are needed in order to reach outcome in convexified set. Then implementation of such equilibrium might require a lot of information and calculation, which in practice is usually not worth to do so. In light of this, one might decide to use lottery to decide the first mover each time instead of our system. the lottery probability is set in the way such that the expected payoff would be the required payoff. No doubt, it is much easier to implement but such kind of lottery would have other problem

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[42] Formally speaking, an agreement $\left\langle \left(x_A^k, x_B^k\right), T_k \right\rangle$ is static efficient if there is no other agreement $\left\langle \left(x_A^h, x_B^h\right), T_h \right\rangle$ in $\{X_h \cup \{d_h\}\} \times [0, \infty]$ such that $\left\langle \left(x_A^h, x_B^h\right), T_h \right\rangle \succ_i \left\langle \left(x_A^k, x_B^k\right), T_k \right\rangle$ for $i = A, B$. A sequence of agreements $(\left\langle \left(\tilde{x}_A^1, \tilde{x}_B^1\right), T_1 \right\rangle, \ldots, \langle (\tilde{x}_A^n, \tilde{x}_B^n), T_n \rangle)$ is dynamic efficient if there is no other sequence of agreements $(\left\langle \left(x_A^1, x_B^1\right), T_1 \right\rangle, \ldots, \langle (x_A^n, x_B^n), T_n \rangle)$ in $\{\{X_k \cup \{d_k\}\} \times [0, \infty]\}_{k=1}^n$ such that $(\left\langle \left(x_A^1, x_B^1\right), T_1 \right\rangle, \ldots, \langle (x_A^n, x_B^n), T_n \rangle) \succ_i (\left\langle \left(\tilde{x}_A^1, \tilde{x}_B^1\right), T_1 \right\rangle, \ldots, \langle (\tilde{x}_A^n, \tilde{x}_B^n), T_n \rangle)$ for $i = A, B$.

[43] The reason is that it is never optimal for proposer to make a Pareto dominated offer. For a rational player, he has no reason not to increase his share without reducing other player's share. Suppose the player i wants to make an offer (x_A, x_B) where there is another offer $(\hat{x}_A, \hat{x}_B)$ such that $(\hat{x}_A, \hat{x}_B) \succ_A (x_A, x_B)$ and $(\hat{x}_A, \hat{x}_B) \sim_B (x_A, x_B)$. There is no reason not to purpose $(\hat{x}_A, \hat{x}_B)$ as the share is increased without increase the risk of being rejected. However, there is sometimes that static efficiency does not hold. For example, suppose player i wants to make an offer (x_A, x_B) where there is another offer $(\hat{x}_A, \hat{x}_B)$ such that $(\hat{x}_A, \hat{x}_B) \sim_A (x_A, x_B)$ and $(\hat{x}_A, \hat{x}_B) \succ_B (x_A, x_B)$. If $\langle (x_A, x_B), t \rangle \succ_B \langle (\hat{x}_A, \hat{x}_B), t+1 \rangle$ is true, then player B would not reject it. In this case, there is no reason why player A would try to make player B better unless player A care fore what player B obtains from the agreement.

that it is not ex-post optimal for the player losing the lottery.

6 Non-convex Example

In previous sections, we have discuss bargaining games in general. This section is devoted to the discussion of two special non-convex cases: players with risk-loving attitude and players exhibiting indomitable behaviour.

I Risk loving players

Suppose the players have risk loving attitude which is modeled by $U(x) = x^a$ and $V(y) = y^b$ where $k > 1$. For simplicity, we assume $a = b = k$ and $r_A = r_B = r$. Hence $U^{-1}(u) = u^{1/k}$ and $V^{-1}(v) = v^{1/k}$. The feasible set would then be

$$X_t = \left\{(u, v) \in \mathbb{R}^2 : u^{1/k} + v^{1/k} \leq \delta^t, v \geq 0, u \geq 0\right\}.$$

and the Pareto frontier function is

$$u = \Omega_A^t = \left(\delta^t - v^{1/k}\right)^k \text{ and } v = \Omega_B^t = \left(\delta^t - u^{1/k}\right)^k.$$

The following graph below shows the feasible set X_0 of the model.

Proposition 6 *In one-stage bargaining game $G(1, \infty)$ with set of agreements $X_k = \{(x, y) : x + y \leq 1\}$ and risking loving agent $U(x) = x^a$ and $V(y) = y^a$, the unique SPE outcome is that player A always offers $(x, 1 - x)$ and player B always offers $(1 - y, y)$.*

$$x = \frac{1 - \delta_B^{1/b}}{1 - \delta_A^{1/a}\delta_B^{1/b}} \text{ and } y = \frac{1 - \delta_A^{1/a}}{1 - \delta_A^{1/a}\delta_B^{1/b}}.$$

Proof. *The unique SPE is in fact the same as the general case ($r_A \neq r_B$ and $k_A \neq k_B$) derived using Shaked and Sutton techniques (see in appendix III). Here we would try to*

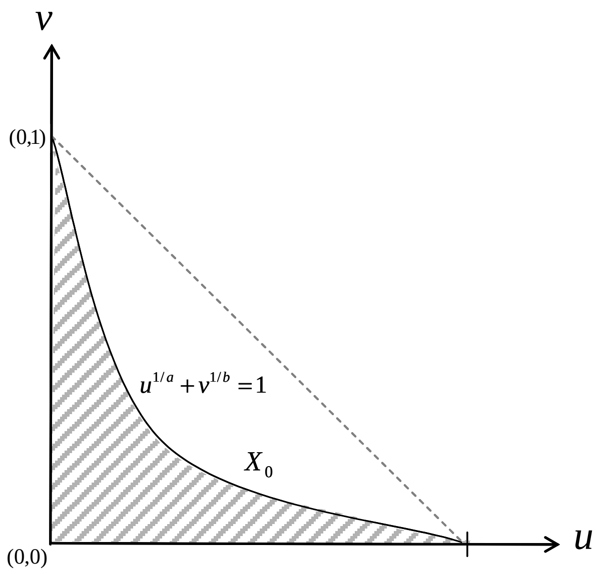

Figure 2: Feasible set for risk loving players

prove the common discount case and common preference using the concept of security equilibrium.

Using the procedure of deletion of iterated conditional dominance, we have, when t is odd ■

$\sup_X E_{0,t}$	$\Omega_A^0(...\Omega_A^{t-3}\left(\Omega_B^{t-2}\left(\Omega_A^{t-1}(0)\right)\right)) = \left[1-\delta^{1/k}+\delta^{2/k}-...+\delta^{(t-1)/k}-0\right]^k$
$\inf_X E_{0,t}$	$\Omega_A^0(...\Omega_A^{t-3}\left(\Omega_B^{t-2}\left(\Omega_A^{t-1}\left(\delta^t\right)\right)\right)) = \left[1-\delta^{1/k}+\delta^{2/k}-...+\delta^{(t-1)/k}-\delta^{t/k}\right]^k$
$\sup_Y E_{0,t}$	$\Omega_B^1(...\Omega_A^{t-3}\left(\Omega_B^{t-2}\left(\Omega_A^{t-1}\left(\delta^t\right)\right)\right)) = \left[\delta^{1/k}-\delta^{2/k}-...-\delta^{(t-1)/k}+\delta^{t/k}\right]^k$
$\inf_Y E_{0,t}$	$\Omega_B^1(...\Omega_A^{t-3}\left(\Omega_B^{t-2}\left(\Omega_A^{t-1}(0)\right)\right)) = \left[\delta^{1/k}-\delta^{2/k}-...-\delta^{(t-1)/k}+0\right]^k$

and when t is even

$\sup_X E_{0,t}$	$\Omega_A^0(...\Omega_B^{t-3}\left(\Omega_A^{t-2}\left(\Omega_B^{t-1}\left(\delta^t\right)\right)\right)) = \left[1-\delta^{1/k}+\delta^{2/k}-...-\delta^{(t-1)/k}+\delta^{t/k}\right]^k$
$\inf_X E_{0,t}$	$\Omega_A^0(...\Omega_B^{t-3}\left(\Omega_A^{t-2}\left(\Omega_B^{t-1}\left(0\right)\right)\right)) = \left[1-\delta^{1/k}+\delta^{2/k}-...-\delta^{(t-1)/k}+0\right]^k$
$\sup_Y E_{0,t}$	$\Omega_B^1(...\Omega_B^{t-3}\left(\Omega_A^{t-2}\left(\Omega_B^{t-1}\left(0\right)\right)\right)) = \left[\delta^{1/k}-\delta^{2/k}-...-\delta^{(t-2)/k}+\delta^{(t-1)/k}-0\right]^k$
$\inf_Y E_{0,t}$	$\Omega_B^1(...\Omega_B^{t-3}\left(\Omega_A^{t-2}\left(\Omega_B^{t-1}\left(\delta^t\right)\right)\right)) = \left[\delta^{1/k}-\delta^{2/k}-...-\delta^{(t-2)/k}+\delta^{(t-1)/k}-\delta^{t/k}\right]^k$

Hence, we have

t	$\sup_X E_{0,t}$	$\inf_X E_{0,t}$	$\sup_Y E_{0,t}$	$\inf_Y E_{0,t}$
odd	$\left[\frac{1+\delta^{t/k}}{1+\delta^{1/k}}\right]^k$	$\left[\frac{1-\delta^{(t+1)/k}}{1+\delta^{1/k}}\right]^k$	$\delta\left[\frac{1+\delta^{t/k}}{1+\delta^{1/k}}\right]^k$	$\delta\left[\frac{1-\delta^{(t-1)/k}}{1+\delta^{1/k}}\right]^k$
even	$\left[\frac{1+\delta^{t+1/k}}{1+\delta^{1/k}}\right]^k$	$\left[\frac{1-\delta^{t/k}}{1+\delta^{1/k}}\right]^k$	$\delta\left[\frac{1+\delta^{(t-1)/k}}{1+\delta^{1/k}}\right]^k$	$\delta\left[\frac{1-\delta^{t/k}}{1+\delta^{1/k}}\right]^k$

It can be easily observed that $\sup_X E_{0,t}$ and $\sup_Y E_{0,t}$ is monotonic decreasing and bounded. Moreover, $\inf_X E_{0,t}$ and $\inf_Y E_{0,t}$ is monotonic increasing and bounded. Hence, the set $E_{0,t}$ collapse as t increases. As t goes to infinity, we could see that

t	$\sup_X E_{0,t}$	$\inf_X E_{0,t}$	$\sup_Y E_{0,t}$	$\inf_Y E_{0,t}$
∞	$\left(\frac{1}{1+\delta^{1/k}}\right)^k$	$\left(\frac{1}{1+\delta^{1/k}}\right)^k$	$\delta\left(\frac{1}{1+\delta^{1/k}}\right)^k$	$\delta\left(\frac{1}{1+\delta^{1/k}}\right)^k$

Hence, we have a unique equilibrium because

$$\begin{aligned}
\text{SPE} &= \bigcap_{t=0}^{\infty} E_{0,t} = \bigcap_{t=0}^{\infty} \left(\{(x,y) : \inf_X E_{0,t} \le x \le \sup_X E_{0,t}, \inf_Y E_{0,t} \le y \le \sup_Y E_{0,t}\} \cap X_0\right) \\
&= \left(\{(x,y) : \inf_X E_{0,\infty} \le x \le \sup_X E_{0,\infty}, \inf_Y E_{0,\infty} \le y \le \sup_Y E_{0,\infty}\} \cap X_0\right) \\
&= \left\{\left(\left(\frac{1}{1+\delta^{1/k}}\right)^k, \left(\frac{\delta^{1/k}}{1+\delta^{1/k}}\right)^k\right)\right\}.
\end{aligned}$$

Caution reader should note that the bargaining outcome between risk-loving agents would be similar to the outcome between risk-neutral agents, except the discount rate changed from δ to $\delta^{1/k}$. This result still holds under general cases. The following proposition summarize the result:

II Indomitable players

Indomitability is rather common in power struggle in which the larger the power is, the larger the desire will be.[44] Another motivation to study indomitability is that psychological and behaviorial effect derived from bargaining outcome. Very often, people would not only just view the outcome negotiation solely based on the absolute amount of benefit but rather, they may have a psychological reservation amount. Usually, the amount reflects the bargainers perception of fair share of bargaining. Players would be dissatisfied if they get less than the fair share of the surplus. Again for simplicity, we would assume common discount factor $r_A = r_B = r$.

Mathematically, the utility function for player A is:

$$u = \begin{cases} \frac{m}{h}x & \text{if } x \leq h \\ \frac{1-m}{1-h}x - \frac{h-m}{1-h} & \text{if } x \geq h \end{cases}$$

and and the utility function for player B is:

$$v = \begin{cases} \frac{n}{k}y & \text{if } y \leq k \\ \frac{1-n}{1-k}y - \frac{k-n}{1-k} & \text{if } y \geq k \end{cases}$$

where $m < h$ and $n < k$ to reflect non-convexity. The following shows how the utility changes with the share of surplus:

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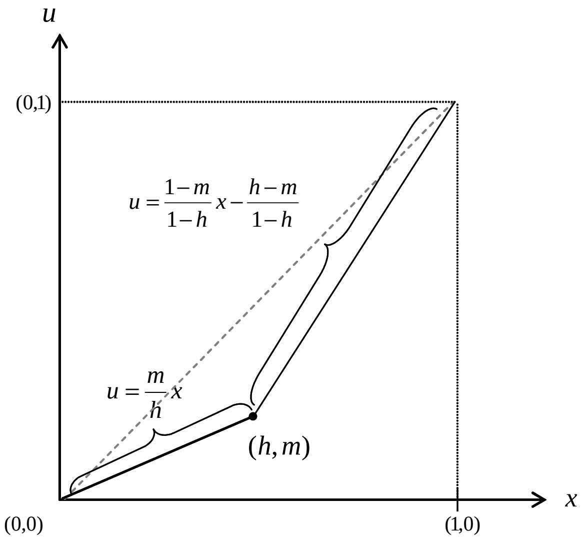
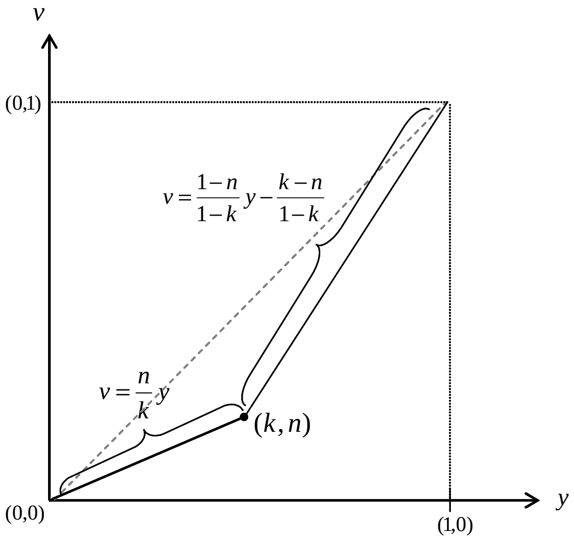
[44] One famous quotation related to this view is "Power corrupts and absolute power corrupts absolutely."



Utility function of player A



Utility function of player B

To obtain the bilinear Pareto frontier, one must set $h = 1-k$ and $h = k$. Hence, we must have $h = k = 1/2$.The cutoff point $h = k = 1/2$ could have a good economic implication. Usually speaking, we consider the fair share value to be one-half of the surplus. There are at least five reasons behind this assumption. Firstly, in many customs and cultures, we know that fair value is somewhere around one-half. Secondly, in power struggle and in the event of impasse, usually the tie-breaking rule is voting, which is determined by rule of majority. Hence, one-half is the critical point. Thirdly, it is mathematical simple to derive Pareto frontier function. If the kink on the utility function is bilinear at one half, the Pareto frontier is bilinear; otherwise, it is multilinear. Fourthly, if we agreed that the most basic non-convex bargaining is the exponential decaying Pareto frontier which is differentiable, the second basic one is bilinear Pareto frontier on is differentiable almost everywhere except the kinked point. Fifthly, this kind of characterization partly catches the concept of relative gain. Therefore the utility functions would change to

$$\begin{cases} u^t = 2mx\delta^t \\ v^t = \left[2(1-m)y - (1-2m)\right]\delta^t \end{cases} \text{if } x \le 1/2, y \ge 1/2$$

and

$$\begin{cases} u^t = \left[2\left(1-m\right)x - \left(1-2m\right)\right]\delta^t \\ v^t = 2my\delta^t \end{cases} \text{if } x \ge 1/2, \ y \le 1/2.$$

The Pareto frontier functions are

$$v = \Omega_B^0 = \begin{cases} -\frac{1-n}{m}u + 1 & \text{if } u \le m \\ -\frac{n}{1-m}u + \frac{n}{1-m} & \text{if } u \ge m \end{cases}$$

and

$$u = \Omega_A^0 = \begin{cases} -\frac{1-m}{n}v + 1 & \text{if } v \le n \\ -\frac{m}{1-n}v + \frac{m}{1-n} & \text{if } v \ge n \end{cases}$$

where $m, n < 0$ and $m + n \le 1$ (non-convex assumption).

Figure 3 shows the feasible set at time 0.Note that the point (m, n) is the kink of the frontier. Under assumption of symmetry condition($m = n$) and common discount factor, the function of Pareto frontier are

$$v^t = \Omega_B^t\left(u^t\right) = \begin{cases} -\frac{1-m}{m}u^t + \delta^t \equiv l_B^t\left(u^t\right) & \text{if } u^t \le \delta^t m \\ -\frac{m}{1-m}u^t + \frac{m}{1-m}\delta^t \equiv h_B^t\left(u^t\right) & \text{if } u^t \ge \delta^t m \end{cases}$$

and

$$u^t = \Omega_A^t\left(v^t\right) = \begin{cases} -\frac{1-m}{m}u_B^t + \delta^t \equiv l_A^t\left(v^t\right) & \text{if } v^t \le \delta^t m \\ -\frac{m}{1-m}v^t + \frac{m}{1-m}\delta^t \equiv h_A^t\left(v^t\right) & \text{if } v^t \ge \delta^t m \end{cases}$$

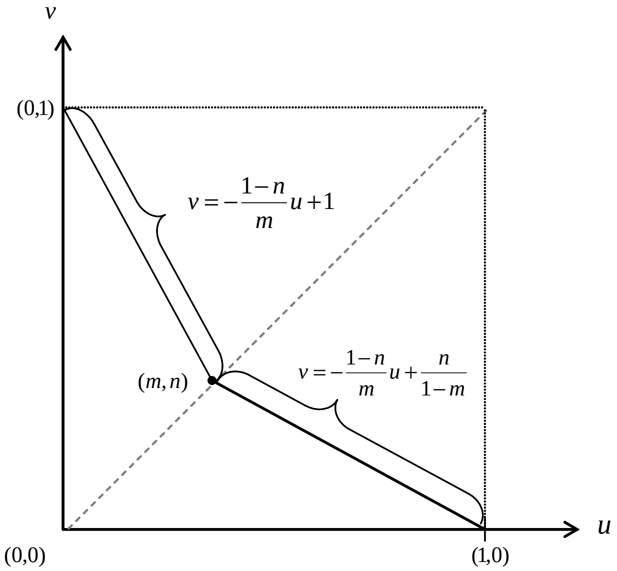


Figure 3: Graph of model with a single-kinked Pareto frontier

which are piece-wise functions. It is not immediately clear that when to use which part. To simplify the complication, suppose $m \leq \delta/(1+\delta)$. This assumption is not too restrictive as players would have high value of δ if they are patient enough or the time lag between offers is short enough.

Proposition 7 *In one-stage bargaining game with set of agreements* $X_k = \{(x,y) : x+y \leq 1\}$ *and indomitable agent, that is, having the following forms of utility:*

$$\begin{cases} u^t = 2mx\delta^t \\ v^t = [2(1-m)y-(1-2m)]\,\delta^t \end{cases} \quad \text{if } x \leq 1/2, y \geq 1/2$$

and

$$\begin{cases} v^t = [2\,(1-m)\,x_A-(1-2m)]\,\delta^t \\ v^t = 2my\delta^t \end{cases} \quad \text{if } x_A \geq 1/2,\ y \leq 1/2.$$

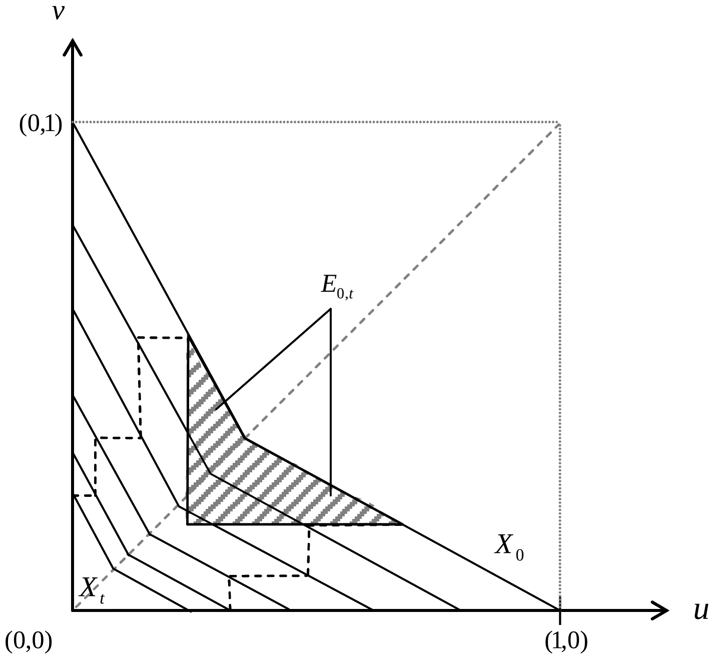


Figure 4: When t is odd

The set SPE outcome is

$$SPE = \left\{ (x,y) : \frac{m}{1-m}\frac{1}{1+\delta} \leq x \leq \frac{1}{1+\delta}, \frac{m}{1-m}\frac{\delta}{1+\delta} \leq y \leq \frac{\delta}{1+\delta} \right\} \cap X_{0.}$$

Proof. We are going to apply the iterated deletion of conditional dominated strategies to solve the SPE. When t is odd, the graphically we have

Therefore, we have, when t is odd (see appendix for derivation)

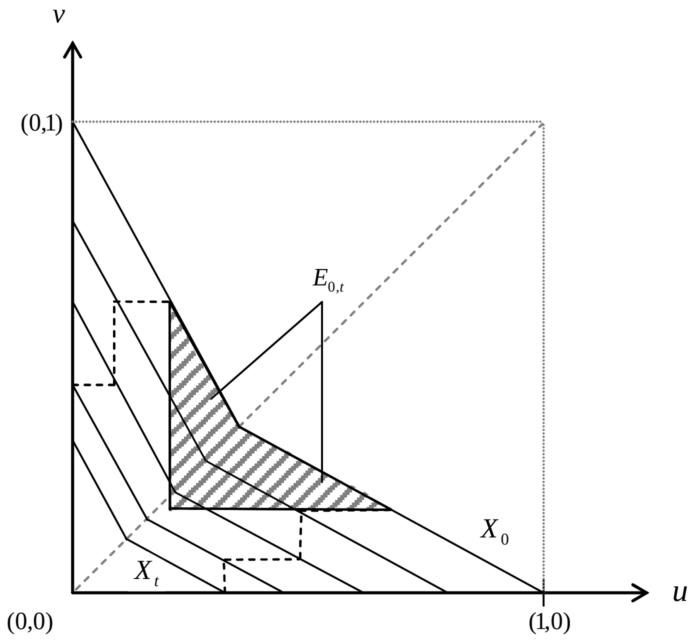


Figure 5: When t is even

t	odd t
$\sup_X E_{0,t}$	$l_A^{t-n}\left(..l_B^{t-4}(h_A^{t-3}\left(l_B^{t-2}\left(h_A^{t-1}(0)\right)\right)\right)) = \dfrac{1+\delta^t}{1+\delta}$
$\inf_X E_{0,t}$	$h_A^{t-n}\left(..h_B^{t-4}(l_A^{t-3}\left(h_B^{t-1}\left(l_A^{t-1}\left(\delta^t\right)\right)\right)\right)) = \dfrac{m}{1-m}\dfrac{1+\delta^t}{1+\delta} - \delta^t$
$\sup_Y E_{t,0}$	$l_B^{t-n+1}\left(..l_A^{t-3}\left(h_B^{t-1}\left(l_A^{t-1}\left(\delta^t\right)\right)\right)\right) = \delta\dfrac{1-\delta^{t-1}}{1+\delta} + \dfrac{1-m}{m}\delta^t$
$\inf_Y E_{t,0}$	$h_B^{t-n+1}\left(..h_A^{t-3}\left(l_B^{t-2}\left(h_A^{t-1}(0)\right)\right)\right) = \dfrac{m}{1-m}\dfrac{\delta\left(1-\delta^{t-1}\right)}{1+\delta}$

and when t is even, graphically we have

and so, we have the following (see appendix for derivation)

t	even t
$\sup_X E_{0,t}$	$l_A^{t-n}\left(..l_A^{t-4}(h_B^{t-3}\left(l_A^{t-2}\left(h_B^{t-1}\left(\delta^t\right)\right)\right)\right)\right) = \frac{1+\delta^{t+1}}{1+\delta}$
$\inf_X E_{0,t}$	$h_A^{t-n}\left(..h_A^{t-4}(l_B^{t-3}\left(h_A^{t-1}\left(l_B^{t-1}\left(0\right)\right)\right)\right)\right) = \frac{m}{1-m}\frac{1-\delta^t}{1+\delta}$
$\sup_Y E_{t,0}$	$l_B^{t-n+1}\left(..l_B^{t-3}\left(h_A^{t-1}\left(l_B^{t-1}\left(0\right)\right)\right)\right) = \frac{\delta\left(1+\delta^{t-1}\right)}{1+\delta}$
$\inf_Y E_{t,0}$	$h_B^{t-n+1}\left(..h_B^{t-3}\left(l_A^{t-2}\left(h_B^{t-1}\left(\delta^t\right)\right)\right)\right) = \frac{m}{1-m}\frac{\delta\left(1-\delta^t\right)}{1+\delta}$

Hence, combining the both cases, we have

t	$\sup_X E_{0,t}$	$\inf_X E_{0,t}$	$\sup_Y E_{0,t}$	$\inf_Y E_{0,t}$
odd n	$\frac{1+\delta^t}{1+\delta}$	$\frac{m}{1-m}\frac{1+\delta^t}{1+\delta}-\delta^t$	$\frac{\delta\left(1-\delta^{t-1}\right)}{1+\delta}+\frac{(1-m)\,\delta^t}{m}$	$\frac{m}{1-m}\frac{\delta\left(1-\delta^{t-1}\right)}{1+\delta}$
even n	$\frac{1+\delta^{t+1}}{1+\delta}$	$\frac{m}{1-m}\frac{1-\delta^t}{1+\delta}$	$\frac{\delta\left(1+\delta^{t-1}\right)}{1+\delta}$	$\frac{m}{1-m}\frac{\delta\left(1-\delta^t\right)}{1+\delta}$

It can be easily observed that $\sup_X E_{0,t}$ and $\sup_Y E_{0,t}$ is monotonic decreasing and bounded. Moreover, $\inf_X E_{0,t}$ and $\inf_Y E_{0,t}$ is monotonic increasing and bounded. Hence, the set $E_{0,t}$ collapse as t increases. As t goes to infinity, we could see that

t	$\sup_X E_{0,t}$	$\inf_X E_{0,t}$	$\sup_Y E_{0,t}$	$\inf_Y E_{0,t}$
∞	$\frac{1}{1+\delta}$	$\frac{m}{1-m}\frac{1}{1+\delta}$	$\frac{\delta}{1+\delta}$	$\frac{m}{1-m}\frac{\delta}{1+\delta}$

From [23], we know the set of SPE is indeed

$$
\begin{aligned}
\text{SPE} &= \bigcap_{t=0}^{\infty} E_{0,t} = \bigcap_{t=0}^{\infty}\left(\{(x,y) : \inf_X E_{0,t} \le x \le \sup_X E_{0,t}, \inf_Y E_{0,t} \le y \le \sup_Y E_{0,t}\} \cap X_0\right) \\
&= \left(\{(x,y) : \inf_X E_{0,\infty} \le x \le \sup_X E_{0,\infty}, \inf_Y E_{0,\infty} \le y \le \sup_Y E_{0,\infty}\} \cap X_0\right) \\
&= \left(\left\{(x,y) : \frac{m}{1-m}\frac{1}{1+\delta} \le x \le \frac{1}{1+\delta}, \frac{m}{1-m}\frac{\delta}{1+\delta} \le y \le \frac{\delta}{1+\delta}\right\} \cap X_0\right)
\end{aligned}
$$

which is shown graphically below ■

Binmore has used non-stationary strategy to support the large set of SPE. For the characterization of the strategy supporting the SPE, please refer to [22] for further details. Before conclusion of this section, we would like to single out four stationary strategies supporting SPE and try to compare them with the standard Rubinstein game. It turns

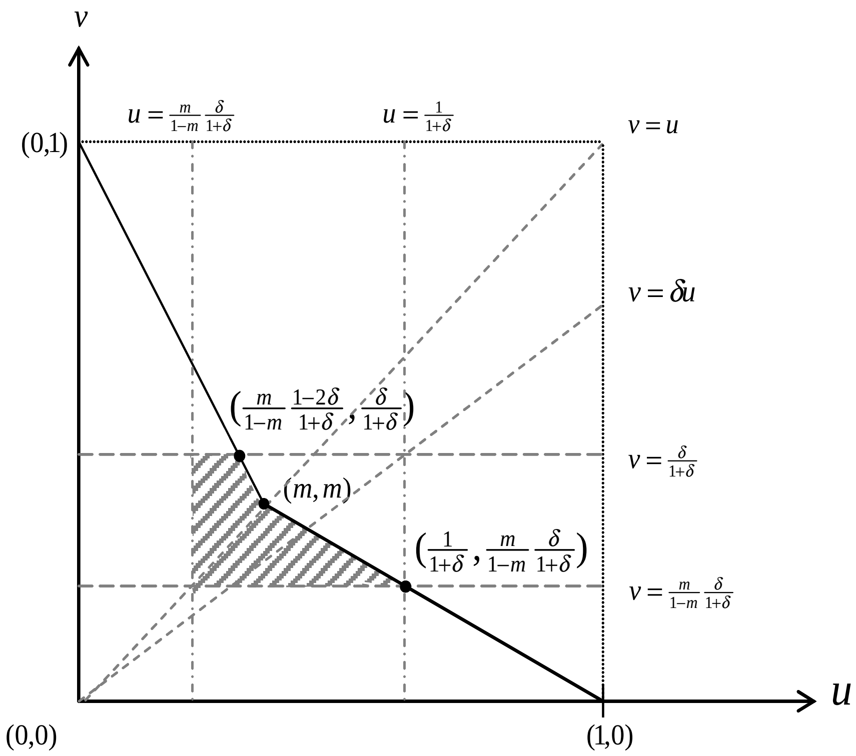


Figure 6: Set of SPE for single-kinked Pareto frontier

out that there are out of the four stationary strategies, two are symmetric and the other two are asymmetric.

Proposition 8 *In one-stage bargaining game with indomitable agent, where utility functions and set of agreements are stated as in previous proposition, there are four stationary strategies* (x, y) *which can be supported as subgame perfect outcome. The four SPE strategies are*

$$\begin{aligned} 1)\ x &= \frac{1}{2(1+\delta)(1-m)} \text{ and } y = \frac{1+(1-2m)+\delta}{2(1+\delta)(1-m)}, \\ 2)\ x &= \frac{1+(1-2m)+\delta}{2(1+\delta)(1-m)} \text{ and } y = \frac{1}{2(1+\delta)(1-m)}, \\ 3)\ x &= y = \frac{2m-2m\delta+\delta}{2[m+\delta(1-m)]}, \\ 4)\ x &= y = \frac{1}{2[m\delta+(1-m)]} \end{aligned}$$

where player i *is to accept any offer having more payoff than* x *and to offer* y *independent of history.*

Proof. *Suppose that player* A *always offers* x *and player* B *always offer* y *such that the payoff for player* A *being the proposer, and the payoff of player* B *being the proposer would always be* $\delta^t u$ *and* $\delta^t v$ *if they has reached agreement in time* t. *Using the stationarity property, we could derive the following results:*

(1) If we have $u \leq m$ *and* $v \geq m$, *we would have*

$$-\frac{1-m}{m}u+1=\delta v \text{ and } -\frac{m}{1-m}u+\frac{m}{1-m}=\delta v$$

which becomes

$$u=\frac{m}{1-m}\frac{1}{1+\delta} \text{ and } v=\frac{1}{1+\delta}.$$

Hence we would have

$$x=\frac{1}{2\left(1+\delta\right)\left(1-m\right)}\text{ and }y=\frac{1+\left(1-2m\right)+\delta}{2\left(1+\delta\right)\left(1-m\right)}$$

(2) And the analysis for $u\geq m$ *and* $v\leq m$ *would also result in same result with (1) but with different subscript.*

(3) Supposing $u\geq m$ *and* $v\geq m$*,*

$$-\frac{m}{1-m}u+\frac{m}{1-m}=\delta v\text{ and }-\frac{m}{1-m}v+\frac{m}{1-m}=\delta u$$

which we would have

$$u=v=\frac{m}{m+\delta\left(1-m\right)}\text{ and }x=y=\frac{2m-2m\delta+\delta}{2\left[m+\delta\left(1-m\right)\right]}.$$

(4) Finally, if $u_A^*\leq m$ *and* $u_B^*\leq m$*, we have*

$$-\frac{1-m}{m}u_A^*+1=\delta u_B^*\text{ and }-\frac{1-m}{m}u_B^*+1=\delta u_A^*$$

which becomes

$$u_A^*=u_B^*=\frac{m}{m\delta+\left(1-m\right)}\text{ where }x_A^*=x_B^*=\frac{1}{2\left[m\delta+\left(1-m\right)\right]}$$

with only restriction $m<\delta$*.* ■

For symmetric equilibrium, if players are so patient or the time lag between offers are minimized that $\delta\rightarrow1$, then $x=y=1/2$ since there should be no first-mover advantage. If we take $m\rightarrow1/2$ which indeed is the standard Rubinstein game, then we have $x=y=1/\left(1+\delta\right)$ which return us the standard bargaining solution. If we collapse the kink point

$m \to 0$, then $x = y = 1/2$. This case is very closed situation of problem of two competitive firms facing a very small market in which mutual entries would have little payoff while monopolization returns a huge profit because the fixed cost is huge. Traditional method to resolve this problem is by collusion or by joint venture. However, it is very likely that the antitrust department would forbid this kind of agreement. In this case, if two firms could have the chance to enter such situation for more than one time, the two firms could use could resolve the inefficiency under repeated bargaining. Intertemporal coordination can be used to ensure a better outcome. In there are many markets two firms might share, they might take turns to share the market. This might be one of the reasons why some cartels appears in the form of locational monopolization in which one firm is assigned to be sole seller in that market.

For asymmetric equilibrium, different from the previous symmetric cases, even if $\delta \to 1$, we still have $x = y \leq 1/2$ which means the asymmetric retains even if discount rate converge to one. This is a very interesting situation because in all convex bargaining, it is impossible to obtain non-symmetric outcome when players become very patient. If the game has status quo with such asymmetric conditions, the game would continue to play as usual as it is not profitable for any player to deviate from the previous condition alone. This might be applied to explain why the bargaining would not usually ends in one-half although it seems it should be the focal point and discount rate is usually converge to one given the small time lag between offers and counteroffers. One could resume the symmetric property only when $m \to 1/2$. By this, we could establish a new view that inequality and conflict might arise due to increasing return to scale or high fixed cost and status quo. Hence, status quo would affect the bargaining outcome which is not possible in convex cases. Different from traditional model, apart from patience, the bargaining power might come from status quo. This might be important in empirical application and our intuition because the classical model simply cannot have the status quo affect the outcome. Another important implication of this result is that in the traditional agenda model, in order to obtain the result that agenda matters, the assumption that utility derived form negotiation only realized when all consensus on all issues reached is needed. The restriction is needed to ensure the outcome from previous issues would affect the next issues. However, it is very rare that consensus can be reached on most issue in most political negotiation. With non-convex model, we would eliminate the need to have such restriction since the game by itself is outcome-dependent. In daily life example, more often than not, household work has high fixed cost and each task is usually performed by one person (usually the mother) due to efficiency reasoning. When the assigned requests the other members of the family to share the burden, it is very difficult to convince other to join the project. This is due to the poor initial status of the assigned.

7 Application

(1) Crisis Bargaining in Hostage Negotiation and Peace Negotiation

In this thesis, we spend a great deal of pages on bargaining stage with predetermined deadline. The literature seems not interested in this kind of modeling. However, in reality, bargaining with deadline indeed is more common than the infinite bargaining. In particular, bargaining with deadline could model the situation that players are eager to solve the problem and would been seriously unhappy if the negotiation continues after certain period of time. For example, in hostage negotiation, both the police and the gang are eager to solve the problem immediately. If the police could not resolve the issue quick enough, then she would be blamed as usefulness and inefficient. If the gang cannot obtain what they have intended quickly, their supplies such as water and food would be inadequate to sustain for long time and the gang would be under huge pressure as police special force might try to resolve the issue through force. Another situation with predefined deadline is political peace talk. The six-party talks between US, Japan, South Korea, Russia, China, North Korea, which aims to find a peaceful solution to security concerns as a result of North Korea nuclear weapons program, has very clear predefined deadline for each round and each phase.

(2) Controlling right in companies and coalition government

We have modeled the non-convex feasible bargaining example: single-kinked utility and Pareto Frontier. This underlying assumption is the marginal return would have sudden rise after certain critical share from surplus. This is evidently true when we are saying that two groups are discussing how to divide a surplus from joint projects. The division scheme could be best represented by the shares in the venture established for cooperation purpose. Clearly, due to the majority rules in modern corporation management, when one group has controlling shares of the company, this group would have extra rights and advantages over the other group.

Political parties joining to form a coalition government also face the same problem of division of power. Usually, there are various commissions being established to deal with various aspects of different issues. Marginal benefit of having one more vote is better when one holding the majority votes than otherwise as the marginal cost to get the minority satisfied is very easy.

(3) Symbolic value from bargaining outcome

Very often, though getting larger share would not have extra tangible benefit, the intangible benefit would be great. This might be having to do with human psychology. In political negotiation over the boundary dispute, clearly there is a reputation effect. For example, if one state is going to get more than, for example, the other country, the people of country getting less than one-half would feel the administration is not doing the good job and the people of the 'winning' country might have extra happiness derived from the 'winning agreement'. Perhaps, people might feel getting less than certain amount is losing face and getting more than certain amount would give a sense of pride. Hence, marginal benefit would increase when the negotiation outcome is greater than some critical value.

(4) Bargaining over Bargaining (on the size of surplus)

Departments try to make use of all budgets, major shareholders and minor shareholders might need to bargain over how much and when the profit is going to be distributed. This kind of model would be closely related to bargaining with inner option.

(5) Alternative Modeling of one-stage bargaining

Rather than have quick solution to the problem, the first phase of negotiation is just to serve as an initial testing and the second phase is the heart of the cooperation. Two-stage bargaining would likely to be with first stage bargaining: finite period bargaining and second stage bargaining: infinite period bargaining (with inside option). Sometimes, mixture of finite period model and infinite model is useful. It can be used to model the time constraint to get an agreement in the first place and subsequent bargaining is to renegotiate for better terms. One such local[45] example is the bargaining between land developer and financial firms. After land developer successfully bided the land, she is required to settle the transaction by cash within 30 days. It is usual practice in Hong Kong that land developer would first finance the lump sum payment by loan from few banks and then refinance the loan through syndicated loans. To model the situation, we model the situation using two stage model. The first stage contains only finite number of period while the second stage is an infinite period bargaining. The discount factor in the second stage is much lower than the first stage because the time constraint to meet the deadline no longer exists.

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[45] Here, local means Hong Kong.

8 Conclusion

In finite stage, finite period models $G(n, z)$, we established the unique stage-and-period stationary SPE. No matter what history is, the proposer would make an offer according to the number of periods remaining in this stage and number of stage in the whole repeated bargaining game. Using idea of backward induction, the path of play is unique and globally stable. Any single deviation by chance or on purpose would not lead to off-equilibrium outcome.

In finite stage, infinite period models $G(n, \infty)$, we established unique stage stationary SPE. Surprisingly, though the whole game can last forever, as each stage by itself is an infinite horizon game, we find out that the strategy for each player in each stage is unique. Players would always propose the same share of the surplus provided that number of remaining stages would not change. It turns out that the more the remaining stages, the lower the offer is. This is because the proposer could exploit the status of purposer fully as any delay would reduce the whole flow of surpluses. The path of play is also unique and globally stable. Any single deviation by chance or on purpose would not lead to off-equilibrium outcome.

In infinite stage, finite period models $G(\infty, z)$ and infinite stage, infinite period models $G(\infty, \infty)$, we have refined the huge set of SPE using globally stable criteria. Though the refinement, we can remove those extremal outcomes which we would think it is impossible in practice. Moreover, if feasible set is non-convex, we could support outcomes that is outside the feasible set. By this, dynamic efficient solution which is beneficial to both parties but unable to obtain in one-shot interaction could be obtained through infinitely repeated interaction. As in prisoner's dilemma, without infinitely repeated interactions, it is not easy to set up a proper punishment and reward scheme to ensure that no one is going to deviate from long-run win-win situation for short-run selfish gain. Table 3 sums up the main result of the four models:

<table>
<tr><th>Stage\Period</th><th>Finite</th><th>Infinite</th></tr>
<tr><td>Finite</td><td>Unique Stage-Period-Stationary SPE</td><td>Unique Stage-Stationary SPE</td></tr>
<tr><td>Infinite</td><td colspan="2">Refine the huge set of SPE using global stability property
Support outcome in the convexified feasible set</td></tr>
</table>

Table 3: Summary of Main Result

For the inter-relationship of the four types of model, it is better to understand by having the stage game being the standard bargaining problem. It can be shown(although it has not shown explicitly in this thesis) that for both finite models, if we allow the finiteness of stage converge to infinity, we could re-obtain the corresponding strategies in the infinite models. (1) For the unique stage-period-stationary strategy, if we allow the period to go infinite, we could obtain the unique stage-stationary strategy in infinite period, finite stage model. (2) For the unique stage-stationary strategy in infinite period, finite stage models, when we take the stage to be infinite, we could re-obtain the uniqueness stationary SPE outcome obtained by Muthoo [2]. (3) It should be noted that however, when we extend our models from finite stage to infinite stage, the uniqueness property disappears as Folk theorem allows huge set of outcomes to be supported as SPE using various punishment and reward schemes.

During the derivation, we also find out a few less important but rather interesting features of the various setup of models. If we allow a finite period, finite stage model to retain proposer status after deadline is reached, we might come across inefficiency outcome. In even period, finite stage model, the proposer is at the disadvantage because acceptance of the proposal leads to an inferior condition due to loss of ultimatum power.

In general, the result for non-convex feasible set would be similar to the outcome in convex feasible set except that the set of SPE is usually huge and not unique. We have analyzed two special cases: (1) Surprisingly, the uniqueness is retained in bargaining with risk loving player; (2) If bargainers have indomitable behaviour, though the whole set of SPE is huge, the number of stationary SPE is limited to four. Out of the four stationary SPE, two are symmetric and the other two are asymmetric. For asymmetric SPEs, the asymmetric feature retains even when the players become sufficiently patient. This is going to imply that asymmetry might be due to status quo if we adjust the model of moving the disagreement point or status quo from origin to some other non-zero points. Particularly, if the game is to be repeated, one could easily deduce that if players only play stationary strategy, a better start would imply a better result. This might be interesting to have further study because in convex model, bargaining outcome is determined by players' patience, bargaining protocol, inside and outside options, commitment tactic. However, in non-convex models, initial sharing condition might have effect in the final outcome due to multiplicity of stationary SPE strategy. Regarding experimental and empirical testing, the inclusion of status quo effect might have significant effect. Another important application of such model might be in political science. In many crisis models framed as war bargaining, the fundamental problem in those models is that while they recognize war is always costly and inefficient, they cannot fully justify why wars indeed break out when side-payment is often available. However, one might easily understand why territorial disputes often ended in war. It is not only because of indivisibility of land but also due to the fact that the weak side has to resort to brutal force to change the status quo.

9 Appendix

I Alternative Model setup: assumption of recognition of the first proposer of each stage

In the paper of Fershtman [8], he has mentioned that if each bargaining bargaining stage is independent of each other, the outcome would be just the same as single-stage bargaining. In the Muthoo's paper [2], the author broke down the independence assumption by assuming the next stage bargaining would start after the conclusion of previous stage with time lag. Although the author had not mentioned why the first proposer of the next stage is the one who accepted other player's proposal in the previous stage, we believe it is designed to show the unique stationary strategy SPE because any player would face the same decision in every period of each stage. This assumption works well for repeated infinite horizon bargaining because there is no deadline. However, for repeated finite horizon bargaining, the assumption of assignment of first proposer of next stage when the previous stage game reaches deadline would directly change the solution.

If we assume no matter the ultimatum is accepted or not, the other player is sure to be the first mover of the next stage, the treatment of equilibrium is simple because the outcome of final period is just proposer-grab-all. If we assume acceptance of ultimatum earns first mover status and rejection leads to lose of the right, we would expected, as we will show, in odd period bargaining, the result is similar to previous cases but for even period bargaining, we would expect there will be strategic delay if the surplus of next stage is to be great. Let me illustrate this using two-period, two-stage bargaining. For simplicity, let the players are risk neutral and sum of offers at stage k must be size π^k. Let $\left(x_t^k\pi^k, \left(1-x_t^k\right)\pi^k\right)$ and $\left(\left(1-y_t^k\right)\pi^k, y_t^k\pi^k\right)$ be the offer made by player A and player B in period t in stage k respectively.

As we know the last stage would be solved uniquely with $x_0^2 = (1 - \delta_B)\,\pi^2$ and $y_0^2 = (1 - \delta_A)\,\pi^2$, in the last period of the first stage, if player A is to accept the proposal y_1^1, the payoff would be $\pi^1 - y_1^1 + \alpha_A (1 - \delta_B)\,\pi^2$ while rejection would have payoff $\alpha_A \delta_A \pi^2$. Hence, the cake would be wasted if

$$\begin{aligned} \alpha_A \delta_A \pi^2 &\leq \pi^1 - y_1^1 + \alpha_A (1 - \delta_B)\,\pi^2 \\ \Rightarrow \quad y_1^1 &\leq \alpha_A (\delta_A + \delta_B - 1)\,\pi^2 - \pi_{B,1}^1. \end{aligned}$$

Then, if $\alpha_A (\delta_A + \delta_B - 1)\,\pi^2 - \pi_{B,1}^1 < 0$, then $y_1^1 < 0$ but player B could not make a negative offer. This condition would happen easily when future surplus is large and players are patient. If this is to be true, player B should accept any offer from player A. In this case, player B could only have $\alpha_B (1 - \delta_A)\,\pi^2$ and rejection only has $\alpha_B \delta_B^2 \pi^2$. Hence, there would be strategic delay if

$$\begin{aligned} \alpha_B \delta_B^2 \pi^2 &> \alpha_B (1 - \delta_A)\,\pi^2 \\ \Rightarrow \quad \alpha_B \left[\delta_B^2 - (1 - \delta_A)\right] &> 0 \end{aligned}$$

which is possible when player B is much more patient that player A.

II Proof of equilibrium for finitely repeated Rubinstein bargaining problem

To illustrate the formal proof, we now show the cases of two stage infinite stage bargaining $G(2,\infty)$. We will employ Shaked and Sutton's technique [7].Denote m_i and M_i be the infimum and supremum of the player i's SPE payoff of the first stage subgames that begins with an offer by player A. Also, let n_i and N_i be the infimum and supremum of the player i's SPE payoff of the first stage subgames that begins with an offer by player B. Note that the second stage must be solved uniquely as standard Rubintein bargaining solution.

First, when player A makes an offer x in the first stage,[46] player B would accept this offer if together with the payoff from this offer with the second stage SPE payoff is larger than supremum of the payoff after one period delay plus the second stage SPE payoff. Hence, we never have

$$1-x+\alpha_B\left(1-\delta_A\right)/\left(1-\delta_A\delta_B\right) > \delta_B M_B + \alpha_B\delta_B^2\left(1-\delta_A\right)/\left(1-\delta_A\delta_B\right)$$

which means

$$m_A \geq 1-\delta_B M_B + \frac{\alpha_B\left(1-\delta_A\right)\left(1-\delta_B^2\right)}{1-\delta_A\delta_B}.$$

Similarly, by symmetric argument, we have

$$m_B \geq 1-\delta_A M_A + \frac{\alpha_A\left(1-\delta_B\right)\left(1-\delta_A^2\right)}{1-\delta_A\delta_B}.$$

Since player B would never offer player A more than $\delta_A M_A$ in the first stage, we never have

$$1-y+\alpha_A\frac{1-\delta_B}{1-\delta_A\delta_B} > \delta_A M_A + \alpha_A\delta_A\frac{\delta_A\left(1-\delta_B\right)}{1-\delta_B\delta_A}$$

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[46] Here, player A makes offer $(x,1-x)$ and player B makes offer $(1-y,y)$

which implies

$$N_A \leq \delta_A M_A - \frac{\alpha_A (1-\delta_B)\left(1-\delta_A^2\right)}{1-\delta_A\delta_B}.$$

Since player B could obtain at least $\delta_B m_B +$ $\alpha_B \delta_B^2 (1-\delta_A)/(1-\delta_A\delta_B)$ if he is to reject A's offer, he would reject x such that

$$1 - x_A + \alpha_B \frac{1-\delta_A}{1-\delta_A\delta_B} \leq \delta_B m_B + \alpha_B \delta_B^2 \frac{1-\delta_A}{1-\delta_A\delta_B}$$

which implies

$$\begin{aligned} M_A &\leq \max\left\{1-\delta_B m_B + \frac{\alpha_B \delta_B^2 (1-\delta_A)}{1-\delta_A\delta_B}, \delta_A N_A\right\} \\ &= \max\left\{1-\delta_B m_B + \frac{\alpha_B \left(1-\delta_B^2\right)(1-\delta_A)}{1-\delta_A\delta_B}, \delta_A^2 M_A - \frac{\alpha_A \delta_A (1-\delta_B)\left(1-\delta_A^2\right)}{1-\delta_A\delta_B}\right\}. \end{aligned}$$

Suppose the latter argument is larger, then we have

$$M_A \leq -\frac{\alpha_A \delta_A (1-\delta_B)}{1-\delta_A\delta_B}$$

which is never possible since payoff is always non-negative. Therefore, we must have the first argument is larger, then we have

$$M_A \leq 1-\delta_B m_B + \frac{\alpha_B \left(1-\delta_B^2\right)(1-\delta_A)}{1-\delta_A\delta_B} \text{ and } M_B \leq 1-\delta_A m_A + \frac{\alpha_A \left(1-\delta_A^2\right)(1-\delta_B)}{1-\delta_A\delta_B}.$$

Then we have

$$m_B \geq \frac{1-\delta_A}{1-\delta_A\delta_B} + \frac{\alpha_A (1-\delta_B)\left(1-\delta_A^2\right) - \alpha_B \delta_A \left(1-\delta_B^2\right)(1-\delta_A)}{(1-\delta_A\delta_B)^2}$$

and

$$M_B \leq \frac{1-\delta_A}{1-\delta_A\delta_B} + \frac{\alpha_A\left(1-\delta_B\right)\left(1-\delta_A^2\right) - \alpha_B\delta_A\left(1-\delta_B^2\right)\left(1-\delta_A\right)}{\left(1-\delta_A\delta_B\right)^2}$$

However $m_B \leq M_B$, this means

$$m_B = M_B = \frac{1-\delta_A}{1-\delta_A\delta_B} + \frac{\alpha_A\left(1-\delta_B\right)\left(1-\delta_A^2\right) - \alpha_B\delta_A\left(1-\delta_B^2\right)\left(1-\delta_A\right)}{\left(1-\delta_A\delta_B\right)^2}$$

Under common discount factor, we have

$$m = M = \frac{1}{1+\delta} + \frac{\alpha\left(1-\delta\right)}{1+\delta},$$

Using the logic, for n stage bargaining, under common discount factor, we have

$$\begin{aligned} x^*_{n,0} &= \frac{1}{1+\delta} + \frac{\alpha\left(1-\delta\right)}{1+\delta} + \frac{\alpha^2\left(1-\delta\right)}{1+\delta} + ... + \frac{\alpha^{n-1}\left(1-\delta\right)}{1+\delta} \\ &= \frac{\pi}{1+\delta}\left[\frac{1-\alpha\delta-\alpha^n\left(1-\delta\right)}{1-\alpha}\right]. \end{aligned}$$

As $n \to \infty$, we would have

$$x^*_{\infty,0} = \frac{\left(1-\delta\alpha\right)}{\left(1+\delta\right)\left(1-\alpha\right)}$$

Note that from Muthoo [2], we know that

$$x^*_{\infty,0} = \frac{\left(1-\delta_j\alpha_j\right)\left(1-\delta_i\alpha_i\right) - \left(\delta_j-\alpha_j\right)\left(1-\delta_i\alpha_i\right)}{\left(1-\delta_j\alpha_j\right)\left(1-\delta_i\alpha_i\right) - \left(\delta_j-\alpha_j\right)\left(\delta_i-\alpha_i\right)}$$

Hence when $r_A = r_B = r$,

$$\begin{aligned} x^*_{\infty,0} &= \frac{(1-\delta\alpha)^2 - (\delta-\alpha)(1-\delta\alpha)}{(1-\delta\alpha)^2 - (\delta-\alpha)^2} \\ &= \frac{(1-\delta\alpha)\left[(1-\delta\alpha)-(\delta-\alpha)\right]}{\left[(1-\delta\alpha)-(\delta-\alpha)\right]\left[(1-\delta\alpha)+(\delta-\alpha)\right]} \\ &= \frac{(1-\delta\alpha)}{\left[(1-\delta\alpha)+(\delta-\alpha)\right]} \\ &= \frac{(1-\delta\alpha)}{(1+\delta)(1-\alpha)} \end{aligned}$$

so, there is no discontinuity in taking it to the limiting case. Now consider the sequence of $\left\{x^*_{0,t}\right\}$

$$\begin{aligned} x^*_{t-1,0} - x^*_{t,0} &= \frac{\pi}{1+\delta}\left[\frac{1-\alpha\delta-\alpha^{n-t+1}(1-\delta)}{1-\alpha}\right] - \frac{\pi}{1+\delta}\left[\frac{1-\alpha\delta-\alpha^{n-t}(1-\delta)}{1-\alpha}\right] \\ &= \frac{\pi}{1+\delta}\frac{1-\delta}{1-\alpha}\alpha^{n-t}(1-\alpha) \\ &= \frac{\alpha^{n-t}(1-\delta)}{1+\delta} > 0 \end{aligned}$$

Therefore, the sequence is monotonic from $1/(1+\delta)$ to $(1-\delta\alpha)/\left[(1+\delta)(1-\alpha)\right]$. This is the reason we draw the Figure 1.

III Proof for general risk loving player

Suppose the players have risk loving attitude which is modeled by $U_A(x) = x^a$ and $V(y) = y^b$ where $a, b > 1$. Given the general assumption $x + y \leq 1$, we have $u^{1/\alpha} + v^{1/\beta} \leq 1$. The Pareto Frontier would then be $v = \left(1 - u^{1/a}\right)^b$. Define M_i and m_i be the supremum and infimum of the player i's payoff obtainable from perfect equilibrium in subgame started with player i. Using the techniques from Shaked and Sutton, we know that

$$\begin{aligned}
M_A &= \left\{1 - \delta_B^{1/b}\left[1 - (\delta_A M_A)^{1/a}\right]\right\}^a, \\
m_A &= \left\{1 - \delta_B^{1/b}\left[1 - (\delta_A m_A)^{1/a}\right]\right\}^a, \\
M_B &= \left\{1 - \delta_A^{1/a}\left[1 - (\delta_B M_B)^{1/b}\right]\right\}^b, \\
m_B &= \left\{1 - \delta_A^{1/a}\left[1 - (\delta_B m_B)^{1/b}\right]\right\}^b
\end{aligned}$$

which can be solved as

$$\begin{aligned}
m_A &= M_A = \left(\frac{1 - \delta_B^{1/b}}{1 - \delta_A^{1/a}\delta_B^{1/b}}\right)^a \text{ and} \\
m_B &= M_B = \left(\frac{1 - \delta_A^{1/a}}{1 - \delta_A^{1/a}\delta_B^{1/b}}\right)^b.
\end{aligned}$$

The partition of the surplus would be

$$x = \frac{1 - \delta_B^{1/b}}{1 - \delta_A^{1/a}\delta_B^{1/b}} \text{ and } y = \frac{1 - \delta_A^{1/a}}{1 - \delta_A^{1/a}\delta_B^{1/b}}.$$

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