Helen Liang Memorial Secondary School (Shatin)

Annual Examination 1998/99

PURE MATHEMATICS

Secondary 6S

Date: 14/6/1999

Max. Marks: 100

Time allowed: 3h (8:45 to 11:45 a.m.)

Class No. _____

Name _____

1. This paper consists of Section A and Section B. Answer BOTH sections.
2. Section A: Answer ALL questions.
3. Section B: Answer any FOUR questions.

FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$$

$$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$$

$$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$$

$$\sin A + \sin B = 2 \sin \frac{A + B}{2} \cos \frac{A - B}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A + B}{2} \sin \frac{A - B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A + B}{2} \cos \frac{A - B}{2}$$

$$\cos A - \cos B = -2 \sin \frac{A + B}{2} \sin \frac{A - B}{2}$$

SECTION A (40 marks)
Answer ALL questions in this section.

1. Suppose
$$0 < u < 3,$$
and
$$u_{n+1} = \frac{12}{1 + u_n}.$$
(a) Prove that $\lim_{n \to \infty} u_n$ exists. (4 marks)
(b) Find $\lim_{n \to \infty} u_n$. (1 mark)

2. If
$$\frac{x^2}{4} + \frac{y^2}{16} = 1,$$
find the tangent line at $(\sqrt{2}, 2\sqrt{2})$. (3 marks)

3. (a) Evaluate
$$\lim_{x \to \frac{\pi}{2}} (\pi - 2x)^{\cos x}.$$
(3 marks)
(b) Evaluate
$$\lim_{n \to \infty} \sum_{i=1}^{n} \left(\frac{i-1}{n^2} \right).$$
(2 marks)

4. Evaluate
$$\int \sin \sqrt{x}\, dx.$$
(4 marks)

5. Evaluate
$$\int x \ln^2 x\, dx.$$
(5 marks)

6. Given that
$$x = 2t - \sin t$$
$$y = 1 + \cos t$$
where $t \in \mathbf{R}$. Find $\frac{dy}{dx}$ and $\frac{d^2 y}{dx^2}$ in terms of t. (4 marks)

7. Evaluate
$$\int_0^\infty e^{-x} \cos bx\, dx.$$
(5 marks)

8. Let $f(x) = x^2 \sin x$. Find
$$f^{(80)}\left(\frac{\pi}{2}\right).$$
(5 marks)

9. (a) Solve the inequality
$$x^2 - 5x + 4 > 0.$$
(1 mark)

   (b) Hence, or otherwise, evaluate
$$\int_0^5 |x^2 - 5x + 4|\, dx.$$
(3 marks)

SECTION B (60 marks)
Answer any FOUR questions from this section.
Each question carries 15 marks.

10. Let
$$f(x) = xe^{-x^2}.$$

   (a) Show that f is an odd function. (2 marks)

   (b) Find $f'(x)$ and $f''(x)$. (2 marks)

   (c) Determine the turning points and inflexion points of $f(x)$. (4 marks)

   (d) Determine the points where
      (i) $f'(x) > 0$
      (ii) $f'(x) < 0$
      (iii) $f''(x) > 0$
      (iv) $f''(x) < 0$
      (2 marks)

   (e) Find, if any, horizontal and vertical asymptotes of $y = f(x)$. (2 marks)

   (f) Sketch the graph of $y = f(x)$. (3 marks)

11. (a) Prove that for all $x > -1$,
$$x \geq \ln(1 + x).$$
(4 marks)

(b) Using (a), or otherwise, prove that

$$1 + \frac{1}{2} + \cdots + \frac{1}{n-1} \geq \ln n \geq \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

(5 marks)

(c) (i) Evaluate

$$\lim_{x \to +\infty} \frac{\ln x}{x}.$$

(ii) Hence, or otherwise, find the value of

$$\lim_{n \to \infty} \frac{1 + \frac{1}{2} + \cdots + \frac{1}{n}}{n}.$$

(6 marks)

12. (a) Using the Mean Value Theorem, or otherwise, prove that

$$1 - x + \frac{x^2}{2} > e^{-x} > 1 - x$$

where $x \geq 0$. (5 marks)

(b) Let $f(x)$ be an odd function. It is given that f is decreasing on the interval $[-b, -a]$, where $b > a > 0$, and $f(x) > 0$ on $[-b, -a]$. Prove that $|f(x)|$ is increasing on the interval $[a, b]$. (4 marks)

(c) Let $g(x)$ be continuous on $[0, 1]$.
(i) Prove that

$$\int_0^{\frac{\pi}{2}} g(\sin x)dx = \int_0^{\frac{\pi}{2}} g(\cos x)dx.$$

(ii) Prove that

$$\int_0^{\pi} xg(\sin x)dx = \frac{\pi}{2} \int_0^{\pi} g(\sin x)dx.$$

(6 marks)

13. (a) Let

$$I_r = \int_0^{\frac{\pi}{2}} \cos^r x\, dx,$$

where $r = 0, 1, 2, \ldots$
(i) Show that $I_r = \frac{r-1}{r} I_{r-2}$.
(ii) Hence evaluate I_{2n} and I_{2n+1} for $n = 0, 1, 2, \ldots$
(6 marks)

(b) Show that for any positive integer m,
$$I_{m+2} \leq I_{m+1} \leq I_m.$$
Hence show that
$$\lim_{m \to \infty} \frac{I_{m+1}}{I_m} = 1.$$
(4 marks)

(c) Using the results of (a) and (b), evaluate
$$\lim_{n \to \infty} \frac{4 \times 4}{3 \times 5} \times \frac{6 \times 6}{5 \times 7} \times \frac{8 \times 8}{7 \times 9} \times \cdots \times \frac{(2n)(2n)}{(2n-1)(2n+1)}.$$
(5 marks)

14. (a) (i) If
$$\frac{1}{x^3 - 1} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1},$$
find the values of A, B and C.

(ii) Hence prove that
$$\int \frac{dx}{x^3 - 1} = \frac{1}{6} \ln \left| \frac{(x-1)^2}{x^2 + x + 1} \right| - \frac{4}{3\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}} \right) + k,$$
where k is an arbitrary constant.
(8 marks)

(b) (i) Prove that
$$\frac{d}{dx}\left(\frac{1}{\sin x \cos x} \right) = \frac{2}{\cos^2 x} - \frac{1}{\sin^2 x \cos^2 x}.$$

(ii) Hence find
$$\int \frac{dx}{\sin^2 x \cos^2 x}.$$
(7 marks)

END OF PAPER