

**Helen Liang Memorial Secondary School (Shatin)**

**Half-yearly Examination 1998/99**

**PURE MATHEMATICS**

Secondary 6S

Date: 12/1/1998

Max. Marks: 100

Time allowed: 2h (8:45 to 10:45 a.m.)

Class No. \_\_\_\_\_

Name \_\_\_\_\_

1. This paper consists of Section A and Section B. Answer BOTH sections.
2. Section A: Answer ALL questions.
3. Section B: Answer ALL questions.

**FORMULAS FOR REFERENCE**

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$$

$$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$$

$$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

**SECTION A (40 marks)****Answer ALL questions in this section.**

1. Prove that for all positive integers
- $n$
- ,

$$5^n + 2 \times 3^{n-1} + 1$$

is divisible by 8. (7 marks)

2. Find the largest coefficient in the expansion of

$$(4 + 3x)^{19}$$

(7 marks)

3. Resolve

$$\frac{(x+3)^2}{x(x+1)(x+2)}$$

into partial fractions. (4 marks)

4. If the roots of

$$x^3 + Hx + G = 0$$

form an arithmetic sequence, prove that  $G = 0$ . (6 marks)

5. Let
- $f(x), g(x)$
- be polynomials, and
- $\beta$
- be a double root of

$$(f(x))^2 + (g(x))^2 = 0.$$

If  $f(x) = 0$  and  $g(x) = 0$  has no common roots, prove that  $\beta$  is a root of the equation  $f'(x)^2 + g'(x)^2 = 0$ . (6 marks)

6. Solve for
- $x$
- if

$$\frac{5x^2 + 13x - 22}{x^2 + 3x - 4} \geq 4$$

(5 marks)

7. Let
- $a, b, c, d$
- be non-zero real numbers. Prove that

$$\sqrt{a+b+c+d} \geq \frac{\sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d}}{2}$$

When does equality hold? (5 marks)

**SECTION B (60)****Answer ALL questions from this section.****Each question carries 20 marks.**

8. (a) Let
- $x_1, x_2, \dots, x_n$
- be an arithmetic sequence.

(i) Prove that

$$\sqrt[n]{x_1 x_2 \dots x_n} \leq \frac{x_1 + x_n}{2}.$$

(ii) Prove that for every  $k = 1, 2, \dots, n$ ,

$$x_k x_{n-k+1} \geq x_1 x_n,$$

hence prove that

$$\sqrt[n]{x_1 \dots x_n} \geq \sqrt{x_1 x_n}.$$

(12 marks)

(b) Hence prove that

$$n^{\frac{n}{2}} < n! < \left(\frac{n+1}{2}\right)^n.$$

(8 marks)

9. (a) Factorize

$$(ab + bc + ca)^3 - a^3 b^3 - b^3 c^3 - c^3 a^3$$

(7 marks)

(b) Use  $(2x + 1)$  as the variable term to express

$$f(x) = 16x^4 + 32x^3 + 64x^2 + 16x + 9.$$

(7 marks)

(c) Transform the equation

$$x^3 + 6x^2 + 5x - 12 = 0$$

into another equation using translation, such that its sum of roots is equal to 0. (6 marks)

10. (a) Prove that, with clear reasoning, if  $a$  and  $b$  are positive real numbers such that  $a^2 < b^2$ , then  $a < b$ . (4 marks)

(b) Prove that if  $k$  is a positive integer, then

$$\sqrt{k+1} + \sqrt{k-1} < 2\sqrt{k}.$$

Hence, deduce that

$$\frac{1}{\sqrt{k}} < \sqrt{k+1} - \sqrt{k-1}$$

(8 marks)

(c) Hence, or otherwise, prove that for any positive integer  $n$ ,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} < \sqrt{n+1} + \sqrt{n} - 1.$$

(8 marks)

**END OF PAPER**