

Helen Liang Memorial Secondary School (Shatin)
Mock Hong Kong Advanced Level Examination 1999/2000

PURE MATHEMATICS PAPER I

Date: 1-3-2000

Max. Marks: 100

Time allowed: 3 hours (8:45 to 11:45 a.m.)

Secondary 7S Name: _____ Class No.: _____

1. This paper consists of Section A and Section B.
2. Answer ALL questions in Section A.
3. Answer any FOUR questions in Section B.

FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$$

$$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$$

$$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

SECTION A (40 marks)**Answer ALL questions in this section.**

1. Find the straight line in the Argand plane containing all the roots of the equation

$$(z + 1)^5 + z^5 = 0,$$

where $z \in \mathbf{C}$. (5 marks)

2. Prove that $2^n > 1 + n\sqrt{2^{n-1}}$ for all $n = 1, 2, \dots$ by considering the arithmetic mean and geometric mean of the numbers $1, 2, 4, \dots, 2^{n-1}$. (6 marks)

3. Let $A = \begin{pmatrix} 1 & 2 \\ -4 & 3 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Evaluate

$$A^2 - 4A + 11I.$$

Hence find A^4 and A^{-1} . (6 marks)

4. (a) By considering the coefficient of x^{n-r} in the expansion $(1+x)^{2n}$, show that

$$C_0^n C_r^n + C_1^n C_{r+1}^n + \dots + C_{n-r}^n C_n^n = \frac{2n(2n-1)\dots(n-r+1)}{(n+r)!}.$$

(4 marks)

- (b) Hence find the value of

$$\sum_{r=0}^{n-1} C_r^n C_{r+1}^n.$$

(4 marks)

5. Given that \vec{a}, \vec{b} are vectors and $|\vec{a}| = 2, |\vec{b}| = 3$ and the angle between \vec{a} and \vec{b} is $\cos^{-1} \frac{3}{5}$. Prove that $\vec{a} - 2\vec{b}$ and $-9\vec{a} + 2\vec{b}$ are perpendicular. (5 marks)

6. Let $\vec{v} = x\hat{i} + y\hat{j}$ be a vector on \mathbf{R}^2 . Find the matrix transformation M such that $M\vec{v}$ gives the projection of \vec{v} onto the line $y = x$. (4 marks)

7. Let $A = \begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix}, B = \begin{pmatrix} 4 & -2 & -1 \\ 4 & 1 & 2 \\ 2 & 2 & -2 \end{pmatrix}$ and $C = \begin{pmatrix} 1 \\ -5 \\ 3 \end{pmatrix}$.

- (a) Evaluate AB and AC . (4 marks)

- (b) Hence solve the equations

$$\begin{cases} 4x - 2y - z = 1 \\ 4x + y + 2z = -5 \\ 2x + 2y - z = 3 \end{cases}$$

(2 marks)

SECTION B (60 marks)

Answer any **FOUR** questions from this section. Each question carries **15 marks**.

8. (a) If $z \in \mathbf{C}$ and $n = 1, 2, \dots$, prove by mathematical induction that, when $z \neq 1$,

$$1 + 2z + 3z^2 + \dots + nz^{n-1} = \frac{1 - (n+1)z^n + nz^{n+1}}{(1-z)^2}.$$

(7 marks)

- (b) Show that, when $z \neq 0$, the right hand side of the identity in part (a) may be written as

$$\frac{z^{-1} - (n+1)z^{n-1} + nz^n}{z^{-1} + z - 2}.$$

Hence, by writing $z = \cos \theta + i \sin \theta$ and using deMoivre's theorem, or otherwise, prove that

$$\begin{aligned} & 1 + 2 \cos \theta + 3 \cos 2\theta + \dots + n \cos(n-1)\theta \\ &= \frac{(n+1) \cos(n-1)\theta - n \cos n\theta - \cos \theta}{2(1 - \cos \theta)}; \end{aligned}$$

where $n = 1, 2, 3, \dots$ (8 marks)

9. Let $p, q \in \mathbf{R}$.

- (a) (i) Show that the system (1) of linear equations

$$(1) \quad \begin{cases} x + y + z = 6 \\ 3x - y + 11z = 6 \\ 2x + y + pz = q \end{cases}$$

in x, y, z has a unique solution if and only if $p \neq 4$.

- (ii) Let $p = 4$ in the system (1). Find all q such that the system (1) has no solution.

(10 marks)

- (b) Show that for all values of p and q , the system (2) of linear equations

$$(2) \quad \begin{cases} x + y + z = 6 \\ 3x + py + 11z = 6 \\ 2x + y + pz = q \end{cases}$$

in x, y, z has a unique solution. (5 marks)

10. Let $A = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$ and let \underline{x} denote a 2×1 matrix.

- (a) Find the two real values λ_1 and λ_2 of λ with $\lambda_1 < \lambda_2$ such that the matrix equation

$$A\underline{x} = \lambda\underline{x}$$

has non-zero solutions. (3 marks)

- (b) Let \underline{x}_1 and \underline{x}_2 be non-zero *general* solutions of $A\underline{x} = \lambda\underline{x}$ corresponding to λ_1 and λ_2 respectively. Show that if

$$\underline{x}_1 = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}$$

and

$$\underline{x}_2 = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$$

then the matrix

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

is non-singular. (4 marks)

- (c) Construct a *particular* X by finding *particular* solutions for \underline{x}_1 and \underline{x}_2 . Using this X , show that

$$AX = X \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and hence

$$A^n = X \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} X^{-1}$$

where n is a positive integer. (5 marks)

- (d) Evaluate

$$\begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}^n.$$

(3 marks)

11. Let $M = \begin{pmatrix} p & q & r \\ r & p & q \\ q & r & p \end{pmatrix}$, where p, q and r are non-negative real numbers satisfying $p + q + r = 1$.

- (a) Show that

$$\begin{aligned} \det(M) &= 1 - 3(pq + qr + rp) \\ &= \frac{(p - q)^2 + (q - r)^2 + (r - p)^2}{2} \end{aligned}$$

Hence deduce that $0 \leq \det(M) \leq 1$. (4 marks)

- (b) Using mathematical induction, or otherwise, show that for any positive integer n , M^n is of the form

$$\begin{pmatrix} p_n & q_n & r_n \\ r_n & p_n & q_n \\ q_n & r_n & p_n \end{pmatrix}$$

where p_n, q_n, r_n are non-negative real numbers satisfying

$$p_n + q_n + r_n = 1.$$

(5 marks)

- (c) Suppose at least two of p, q and r are non-zero. Using (a) and (b), or otherwise, show that
- (i) $\lim_{n \rightarrow \infty} \det(M^n) = 0$.
 - (ii) $\lim_{n \rightarrow \infty} (3p_n - (p_n + q_n + r_n)) = 0$.
- (6 marks)
12. (a) Given a polynomial $P(x)$ with real coefficients, show that if α is a root of $P(x) - x = 0$, then α is also a root of
- $$P(P(x)) - x = 0.$$
- (3 marks)
- (b) Let $P(x) = x^2 + ax + b$, where $a, b \in \mathbf{R}$.
- (i) Using (a), or otherwise, resolve $P(P(x)) - x$ into two quadratic factors.
 - (ii) Find a relation between a and b which is a necessary and sufficient condition for all roots of $P(P(x)) - x = 0$ to be real.
- (8 marks)
- (c) Using (b)(i), or otherwise, solve the equation
- $$(x^2 - 3x + 1)^2 - 3(x^2 - 3x + 1) + 1 - x = 0.$$
- (4 marks)
13. (a) Let x_1, x_2, \dots, x_n be positive real numbers such that

$$\sum_{k=1}^n x_k^2 = \sqrt[3]{na^2}.$$

Prove that

$$\sum_{k=1}^n x_k^3 \geq |a|.$$

- (8 marks)
- (b) Hence, or otherwise, prove that

$$\left(1 + \sqrt[3]{4} + \sqrt[3]{9} + \dots + \sqrt[3]{n^2}\right)^3 \leq \frac{1}{4}n^3(n+1)^2.$$

(7 marks)

END OF PAPER