Helen Liang Memorial Secondary School (Shatin)
Mock Hong Kong Advanced Level Examination 1999/2000

PURE MATHEMATICS PAPER II

Max. Marks: 100
Date: 1-3-2000 Time allowed: 3 hours (1:00 to 4:00 p.m.)

Secondary 7S Name: ______________________ Class No.: _____

1. This paper consists of Section A and Section B.
2. Answer ALL questions in Section A.
3. Answer any FOUR questions in Section B.

FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$
$$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$$
$$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$$
$$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$$
$$\sin A + \sin B = 2 \sin \frac{A + B}{2} \cos \frac{A - B}{2}$$
$$\sin A - \sin B = 2 \cos \frac{A + B}{2} \sin \frac{A - B}{2}$$
$$\cos A + \cos B = 2 \cos \frac{A + B}{2} \cos \frac{A - B}{2}$$
$$\cos A - \cos B = -2 \sin \frac{A + B}{2} \sin \frac{A - B}{2}$$

SECTION A (40 marks)
Answer ALL questions in this section.

1. Suppose that the function f satisfies
$$f(\alpha+\beta)=f(\alpha)+f(\beta)$$
for all $\alpha, \beta \in \mathbf{R}$ and f is continuous at 0. Prove that f is continuous at x for all $x \in \mathbf{R}$. (5 marks)

2. Prove that the equation $x^7+x^5+x^3+1=0$ has exactly one real root. (4 marks)

3. Consider the lines
$$L_1 : \frac{x+2}{-1}=\frac{y-2}{2}=\frac{z}{2}$$
$$L_2 : x+3=0=3y+2z-18$$
(a) Prove that L_1 and L_2 are non-coplanar. (3 marks)
(b) S is a sphere with centre at $C(3,-1,4)$ and touching the line L_1.
(i) Find the radius of the sphere.
(ii) Hence deduce that for any point $P(x,y,z)$ on S,
$$(x-3)^2+(y+1)^2+(z-4)^2=49.$$
(4 marks)

4. Figure 1 shows the graph of $r=a\sin^3\frac{\theta}{3}$, $(a>0)$. Find the area of the shaded region. (6 marks)

FIGURE 1

5. (a) Evaluate $\int_0^1 e^{\sqrt{x}}dx$.
(b) Evaluate $\int_0^\infty e^{-x}\sin x dx$.
(6 marks)

6. Let $f(x)=\frac{1}{\sqrt{1+x^2}}$ for all $x \in \mathbf{R}$. Let $f^{(n)}$ denote the n-th derivative of f for $n=1,2,\ldots$, and $f^{(0)}=f$.
(a) Prove that $(1+x^2)f'(x)+xf(x)=0$. (2 marks)
(b) Hence, or otherwise, evaluate $f^{(2n-1)}(0), n=1,2,\ldots$. (5 marks)

7. Evaluate

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n(n+1)}}+\cdots+\frac{1}{\sqrt{n(2n-1)}}\right).$$

(5 marks)

SECTION B (60 marks)

Answer any FOUR questions from this section. Each question carries 15 marks.

8. Define

$$f(x)=\frac{1}{x(x+1)^2}.$$

(a) Write down the domain of f. (2 mark)

(b) Determine all x such that

(i) $f'(x)>0$

(ii) $f'(x)<0$

(iii) $f''(x)>0$

(iv) $f''(x)<0$

(5 marks)

(c) Find all the relative maxima, relative minima, and inflexion points of $f(x)$. (2 marks)

(d) Find all the vertical asymptote(s) and horizontal asymptote(s) of $f(x)$, if any. (3 marks)

(e) Sketch the graph of $y=f(x)$. (3 marks)

9. (a) Prove that the equation $x=2+\ln x$ has two unequal positive real roots. (5 marks)

(b) The sequence $\{x_n\}$ is defined by

$$x_{n+1}=2+\ln x_n,$$

where $n=1,2,3,\ldots$, and $a<x_1<b$ where a,b are the roots of the equation $x=2+\ln x$. By mathematical induction, or otherwise, prove that

(i) $\{x_n\}$ is an increasing sequence.

(ii) $\{x_n\}$ is bounded above by b.

(7 marks)

(c) Show that $\lim_{n\to\infty} x_n = b$. (3 marks)

10. (a) Let

$$S_n=1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}.$$

Prove that

(i) $\{S_n\}$ is an increasing sequence, i.e. $S_n<S_{n+1}$.

(ii) $\{S_n\}$ is bounded; in fact, $S_n<3$

(3 marks)

(b) Let

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right).$$

Define the exponential function e^x as

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \ldots.$$

Prove that if α is any fixed real number then

$$\lim_{x\to\infty} x^\alpha e^{-x} = 0.$$

(6 marks)

(c) Determine the constants A, B, C and D such that

$$r^3 \equiv A(r-1)(r-2)(r-3) + B(r-1)(r-2) + C(r-1) + D.$$

Hence, or otherwise, evaluate

$$\lim_{n\to\infty}\left(1^3 + \frac{2^3}{1!} + \frac{3^3}{2!} + \cdots + \frac{n^3}{(n-1)!}.\right)$$

(6 marks)

11. (a) Let $f : [1, +\infty) \to [0, +\infty)$ be a monotonic decreasing non-negative function.

(i) Prove that for any $n = 1, 2, 3, \ldots,$

$$f(n+1) \le \int_n^{n+1} f(x)dx \le f(n).$$

(ii) Let

$$a_n = \sum_{k=1}^{n} f(k) - \int_1^n f(x)dx,$$

where $n = 1, 2, 3, \ldots$. Show that the sequence $\{a_n\}$ is convergent.

(9 marks)

(b) Hence, or otherwise, deduce that the limit

$$\lim_{n\to\infty}\left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1} - \ln n\right)$$

exists. (6 marks)

12. (a) Let

$$I_{(p,q)} = \int_a^b (x-a)^p(b-x)^q dx$$

where $b > a$, show that, if $n = 1, 2, \ldots,$

(i) $I_{(n,n-1)} = I_{(n-1,n)}$;

(ii) $2(2n+1)I_{(n,n)} = 2n(b-a)I_{(n,n-1)} = n(b-a)^2 I_{(n-1,n-1)}$.

(7 marks)

(b) Deduce the value of $I_{(n,n)}$ when n is a positive integer. (5 marks)

(c) Hence, or otherwise, evaluate

$$\int_{-1}^{1} (1 - x^2)^6 dx.$$

(3 marks)

13. Consider the parabola

$$\Gamma : y^2 = 4ax,$$

where $a > 0$. Γ is rotated anticlockwise about the origin O through an acute angle θ. In its rotated position, the parabola intersects the y-axis at Q. Let R be the point on the parabola which is symmetrical to Q with respect to the axis of the parabola, L.

FIGURE 2

(a) Find the equation of Γ in its rotated position. (4 marks)

(b) Hence show that the coordinates of R is

$$(8a \sin\theta \tan\theta, 4a \sin\theta(\tan^2\theta - 1)).$$

(6 marks)

(c) Find the value of θ when the y-coordinate of the point R attains its minimum value. (5 marks)

END OF PAPER