Pure Mathematics Paper II

Pre-mock

Section B

1. Let

$$f(x) = \frac{\sin(\pi x)}{x+1}.$$

   (a) Show that

$$\int_a^b f(x)dx = -\int_{a+1}^{b+1} \frac{\sin(\pi x)}{x}dx,$$

   where $a, b \geq 0$. (3 marks)

   (b) Show that for any integer $k \geq 0$,

$$\int_{2k}^{2k+1} f(x)dx > -\int_{2k+1}^{2k+2} f(x)dx > \int_{2k+2}^{2k+3} f(x)dx.$$

   (5 marks)

   (c) Using the results in (b), show that for any integer $n > 1$,

$$0 < \int_0^n f(x)dx < \int_0^1 f(x)dx.$$

   (Hint: Consider separately the cases when n is even and n is odd.) (7 marks)

2. (a) Find

$$\int \frac{x^2}{1+x^2}dx.$$

   Hence find

$$\int \log_e(1+x^2)dx.$$

   (5 marks)

   (b) Let n be a positive integer. Show that

$$\log_e\left(\frac{1}{n^2}\prod_{i=1}^{n}(n^2+i^2)^{\frac{1}{n}}\right) = \sum_{i=1}^{n}\frac{1}{n}\log_e\left(1+\left(\frac{i}{n}\right)^2\right).$$

   Hence evaluate

$$\lim_{n\to\infty}\left(\frac{1}{n^2}\prod_{i=1}^{n}(n^2+i^2)^{\frac{1}{n}}\right).$$

   (10 marks)