1996 Paper 2 Question 10

a. Let $f(x)$ be a function such that $f'(x)$ is strictly decreasing for $x > 0$

(i) Using the Mean Value Theorem, or otherwise, show that

$$f'(k+1) < f(k+1) - f(k) < f'(k)$$

for $k \geq 1$.

(ii) Using (i), show that for any integer $n \geq 2$,

$$f'(2) + f'(3) + \cdots + f'(n) < f(n) - f(1) < f'(1) + f'(2) + \cdots + f'(n-1).$$

(5 marks)

b. Define $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ for any positive integer n.

(i) Using (a), or otherwise, show that

$$H_n - 1 < \ln n < H_n - \frac{1}{n}$$

for $n \geq 2$. Hence, evaluate $\lim_{n\to\infty} \left(\frac{H_n}{\ln n}\right)$.

(ii) Define $\gamma_n = H_n - \ln n$. Show that $\{\gamma_n\}$ is a decreasing sequence and $\lim_{n\to\infty} \gamma_n$ exists. (10 marks)

(15 marks)