THE HONG KONG POLYTECHNIC UNIVERSITY

Department of Applied Mathematics

Programme: BEng(Hons)/MEng in Electrical Engineering (41070)
BEng(Hons) in Electrical Engineering (41076)
BEng(Hons)/MEng in Electrical Engineering (41078)
BEng(Hons) in Electrical Engineering (41079)
BEng(Hons)/MEng in Electronic and Information Engineering (42070)
BEng(Hons)/MEng in Mechanical Engineering (43078)
BEng(Hons)/MEng in Industrial and Systems Engineering (45085)

Subject Code: AMA201 Subject Title: Mathematics I

Session: Semester 1, 2003/2004

Date: 11 December 2003 Time: 6:30 -9:30 pm

Time Allowed: 3 Hours

This question paper has 7 pages (attachments included).

Instructions to Candidates: This question paper has SIX questions.
Attempt FIVE questions.
All questions carry equal marks.

Attachments: Table of Laplace Transformation
Table of Integrals

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1. (a) Consider the following system of linear equations

$$\begin{cases} x_1 \quad\quad\quad\quad\quad - x_3 = p \\ -x_1 + 2x_2 + (p+1)x_3 = 1 \\ px_1 + 2x_2 \quad\quad\quad\quad = 1 \\ -x_1 + 2x_2 \quad\quad\quad\quad = 1 \end{cases},$$

where p is a scalar. Find the values of p for which the system is

(i) consistent with one and only one solution;

(ii) consistent with infinitely many solutions;

(iii) inconsistent.

Find all solution of the system when it is consistent.

[10 marks]

(b) Let $A = \begin{bmatrix} 1 & a^2 & a \\ a & 1 & a^2 \\ a^2 & a & 1 \end{bmatrix}$ where a is a real number. Calculate $\det(A)$ and determine the values of a when A is nonsingular. Use Cramer's rule to solve the system of linear equations $Ax = [1 \quad -1 \quad 0]^T$ when $a = 2$.

[10 marks]

2. (a) Let $A = \begin{bmatrix} 0 & 2 & -3 \\ 2 & 3 & -6 \\ -1 & -2 & 2 \end{bmatrix}$. Find the eigenvalues and corresponding eigenvectors of A. Hence, find a nonsingular matrix P and a diagonal matrix D such that $PDP^{-1} = A$.

[14 marks]

(b) Use the result of Part (a) to find the general solution of the system of linear differential equations

$$\begin{cases} y_1' = 2y_2 - 3y_3 \\ y_2' = 2y_1 + 3y_2 - 6y_3 \\ y_3' = -y_1 - 2y_2 + 2y_3 \end{cases}$$

[6 marks]

3. (a) (i) Find constants A, B, C and D so that

$$y_p(x) = (Ax+B)\cos 2x + (Cx+D)\sin 2x$$

is a particular solution of the differential equation

$$\frac{d^2y}{dx^2} + y = 3x\cos 2x.$$

Hence, obtain the general solution of the differential equation.

(ii) Describe, without giving details, how to find a particular solution of the equation

$$\frac{d^2y}{dx^2} + 4y = 3x\cos 2x.$$

[10 marks]

(c) By the method of *variation of parameters*, find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = \frac{e^{-2x}}{x^2}, \quad x > 0.$$

[10 marks]

4 (a) (x, y) and (r, θ) are related by the equations $x = r\cos\theta$ and $y = r\sin\theta$.

(i) Calculate $\frac{\partial x}{\partial r}$, $\frac{\partial y}{\partial r}$, $\frac{\partial x}{\partial \theta}$ and $\frac{\partial y}{\partial \theta}$.

(ii) Use implicit differentiation to calculate $\frac{\partial r}{\partial x}$, $\frac{\partial \theta}{\partial x}$, $\frac{\partial r}{\partial y}$ and $\frac{\partial \theta}{\partial y}$.

(iii) Calculate the product $\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{bmatrix}$.

[7 marks]

(b) If $w = f(u)$ is a C^2- function and $u = x-t$, use the chain rule of differentiation to show that $w = f(x-ct)$ satisfies $\dfrac{\partial^2 w}{\partial t^2} = c^2 \dfrac{\partial^2 w}{\partial x^2}$.

[6 marks]

(c) Find all critical points of the function

$$f(x,y) = 2x^3 + (x-y)^2 - 6y$$

and determine their nature.

[7 marks]

5. (a) Solve the first order differential equation

$$\frac{dy}{dx} + \left(\frac{3}{x}\right)y = \frac{x}{x^5+1}\ , \ x > 0.$$

[6 marks]

(b) Use Laplace Transform to find the solution of the initial value problem

$$y'' + 3y' - 10y = 3t$$

given that $y(0) = -1$ and $y'(0) = 1$.

[10 marks]

(c) Show that the following differential equation

$$(3x^2y^2 + x + y)\frac{dy}{dx} + (x^3 + y + 2xy^3) = 0$$

is exact. Hence, find its general solutions.

[4 marks]

6. (a) The mass of a fluid passing through a pipe of radius R and length L per unit time satisfies the equation

$$Q = \frac{\pi P R^4}{5VL},$$

where V is the viscosity of the fluid and P is the pressure difference between the two ends of the pipe. It is known that P, R, V and L are measured with relative errors not exceeding $0.5\%, 0.25\%, 0.15\%$ and 0.3% respectively. Using total differentials, find the maximum relative error in the calculated value of Q.

[6 marks]

(b) w is defined as a function of x and y by the equation

$$2xy^2 - x + yw + \cos w = 0.$$

Use implicit differentiation to compute $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$ and $\frac{\partial^2 w}{\partial y \partial x}$.

[8 marks]

(c) A playground is composed of a rectangle surmounted by a semi-circle, as shown in Figure (1). It is required that the playground has an area of 500 square meters. Use the Lagrange Multiplier Method to find its dimensions if its perimeter is to be minimized

[6 marks]

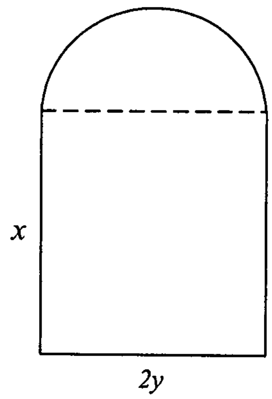

Figure (1)

*** End ***

Laplace Transformation

Definition: $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$.

Useful Formulae:

(1) First derivatives: $\mathcal{L}\{f'(t)\} = s\,F(s) - f(0)$;

(2) Second derivatives: $\mathcal{L}\{f''(t)\} = s^2 F(s) - s\,f(0) - f'(0)$;

(3) Exponential multiplier: $\mathcal{L}\left\{e^{at} f(t)\right\} = F(s-a)$;

(4) Integral: $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \dfrac{F(s)}{s}$.

Table of Laplace Transform

$f(t)$	$F(s)$
1	$\dfrac{1}{s}$
e^{at}	$\dfrac{1}{s-a}$ for $s > a$
$\dfrac{e^{at}-e^{bt}}{a-b}$, for $a \neq b$	$\dfrac{1}{(s-a)(s-b)}$
$\dfrac{a\,e^{at}-b\,e^{bt}}{a-b}$, for $a \neq b$	$\dfrac{s}{(s-a)(s-b)}$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
$\dfrac{1}{(b^2-a^2)}\left(\dfrac{\sin at}{a}-\dfrac{\sin bt}{b}\right)$, for $a^2 \neq b^2$	$\dfrac{1}{(s^2+a^2)(s^2+b^2)}$
$\dfrac{1}{(b^2-a^2)}\left(\cos at - \cos bt\right)$, for $a^2 \neq b^2$	$\dfrac{s}{(s^2+a^2)(s^2+b^2)}$
t^n, n being a positive integer	$\dfrac{n!}{s^{n+1}}$
$\dfrac{t\sin \omega t}{2\omega}$	$\dfrac{s}{(s^2+\omega^2)^2}$
$\dfrac{\sin \omega t - \omega t \cos \omega t}{2\omega^3}$	$\dfrac{1}{(s^2+\omega^2)^2}$
$e^{-at}\sin \omega t$	$\dfrac{\omega}{(s+a)^2+\omega^2}$

Table of Standard Integrals

	$f(x)$	$\int f(x)\,dx$
1	$x^n,\ n \neq -1$	$\dfrac{x^{n+1}}{n+1}$
2	$\dfrac{1}{x}$	$\ln\lvert x\rvert$
3	e^{ax}	$\dfrac{1}{a}e^{ax}$
4	$\sin x$	$-\cos x$
5	$\cos x$	$\sin x$
6	$\tan x$	$-\ln\lvert\cos x\rvert = \ln\lvert\sec x\rvert$
7	$\cot x$	$\ln\lvert\sin x\rvert$
8	$\sec x$	$\ln\lvert\sec x + \tan x\rvert$
9	$\csc x$	$-\ln\lvert\csc x + \cot x\rvert$
10	$\sec x \tan x$	$\sec x$
11	$\sec x \tan x$	$-\csc x$
12	$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$
13	$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{a+x}{a-x}\right\rvert$, for $\lvert x\rvert < a$
14	$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$
15	$\dfrac{1}{\sqrt{a^2+x^2}}$	$\ln\left\lvert x+\sqrt{x^2+a^2}\right\rvert$
16	$\dfrac{1}{\sqrt{x^2-a^2}}$	$\ln\left\lvert x+\sqrt{x^2-a^2}\right\rvert$