THE HONG KONG POLYTECHNIC UNIVERSITY

Department of Applied Mathematics

Programme: BEng(Hons)/MEng in Electrical Engineering (41070)
BEng(Hons) in Electrical Engineering (41076)
BEng(Hons)/MEng in Electrical Engineering (41078)
BEng(Hons)/MEng in Electronic and Information Engineering (42070)
BEng(Hons)/MEng in Mechanical Engineering (43078)
BEng(Hons) in Mechanical Engineering (43088)
BEng(Hons) in Mechanical Engineering (43091)
BEng(Hons) in Manufacturing Engineering (45084)
BEng(Hons)/MEng in Industrial and Systems Engineering (45085)

Subject Code: **AMA201** Subject Title: Mathematics I

Session: Semester 1, 2001/2002

Date: 18 December 2002 Time: 6:30 p.m. - 9:30 p.m.

Time Allowed: 3 Hours

This question paper has 6 pages (attachments included).

Instructions to Candidates: This question paper has SIX questions.

Attempt FIVE questions.

All questions carry equal marks.

Attachments: Table of Laplace Transformation

Table of Integrals

Subject Examiners: Dr. C.K. Chan

Dr. C.W. Chan

Dr. W.K. Chan

Dr. L.K. Li

Dr. A. Loh

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1. Consider the linear system $A\,x = b$,

$$\text{where } A = \begin{bmatrix} 1 & 1 & 1 & k \\ 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & 1 \end{bmatrix},\ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ and } b = \begin{bmatrix} 0 \\ k \\ 1 \\ k^2+2k-1 \end{bmatrix}.$$

(a) Reduce the augmented matrix $B = [A \mid b]$ of the system to its echelon form using only elementary row operations and exact arithmetic.

[10 marks]

(b) Find the values of k for which the system has

(i) a unique solution,

(ii) an infinite number of solutions,

(iii) no solution.

[6 marks]

(c) Also, find the corresponding solution of the system for $k = 0$.

[4 marks]

2. (a) Let $A = \begin{bmatrix} 4 & -1 & -1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

Find all the eigenvalues and corresponding eigenvectors of A. Find a non-singular matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D}$ where $\boldsymbol{D}$ is a diagonal matrix.

[15 marks]

(b) Let $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ and $f(\lambda) = \lambda^2 + \alpha\lambda + \beta$ be the characteristic polynomial of A.

(i) Find α and β.

(ii) Show that $f(A) = A^2 + \alpha A + \beta I = \boldsymbol{0}$.

(iii) Hence, calculate A^{-1}.

[5 marks]

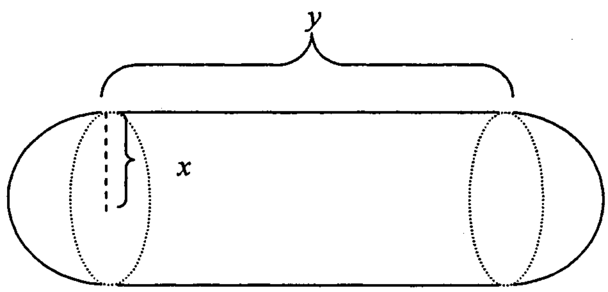
3. (a) Find all the critical points of the function

$$f(x,y) = 240y - 36x + 3x^2 + 2x^3 - 39y^2 + 2y^3.$$

Also, determine the nature of these critical points.

[12 marks]

(b) A container is formed by attaching two hemispheres to the two ends of a cylinder as shown in the following figure.



The container is to hold 18 m^3 of hot liquid. The surfaces of the two hemispheres are made of material costing \$2 per m^2, while the material for the lateral surface of the cylinder costs \$1 per m^2. By means of the method of Lagrange multiplier, find the dimensions of the container that costs the least to make. [Hint: The volume and surface area of a sphere, with radius a are given by $V = \frac{4}{3}\pi a^3$ and $A = 4\pi a^2$ respectively.]

[8 marks]

4. (a) Use total differential to approximate $\dfrac{\sqrt[3]{126}}{\sqrt{15.8}}$. [6 marks]

(b) Suppose $w = g(u)$ is a twice differentiable function of u. If $u = x^2 - 4y$, use the chain rule of differentiation to express $\dfrac{\partial w}{\partial x}$ and $\dfrac{\partial w}{\partial y}$ in terms of $g'(u)$.

Also evaluate $4\dfrac{\partial^2 w}{\partial x^2} - x^2\dfrac{\partial^2 w}{\partial y^2} + 2\dfrac{\partial w}{\partial y}$. [7 marks]

(c) Let w be defined implicitly as a function of x and y by the equation

$$x^2y^3 - w\sin w + 6y^2 = 0.$$

Use implicit differentiation to find $\dfrac{\partial w}{\partial x}$, $\dfrac{\partial w}{\partial y}$ and $\dfrac{\partial^2 w}{\partial x \partial y}$.

[7 marks]

5. (a) Show that the equation

$$(2xy - 2y^2 + 1)\, dx + (x^2 - 4xy - 3)\, dy = 0$$

is exact. Hence find the general solution of the equation.

[7 marks]

(b) Find the general solution of the linear equation

$$x\frac{dy}{dx} + 3y = 2x - \frac{1}{x} + 3.$$

[6 marks]

(c) Solve the following initial value problem using Laplace transform:

$$\begin{cases} \dfrac{dx}{dt} = 3x + 4y \\ \dfrac{dy}{dt} = x + 3y \end{cases}$$

with $x(0) = 0$ and $y(0) = 1$. [7 marks]

6 (a) Solve the differential equation

$$y'' + 4y' + 4y = e^{-t},$$

by means of Laplace transforms, given that $y(0) = y'(0) = 0$.

[10 marks]

(b) Use the method of undetermined coefficients to find a particular solution of the second order differential equation

$$y'' - 5y' + 6y = 2e^{-2x} + 3x - 1.$$

Also, determine the general solution of the equation.

[10 marks]

*** End ***

Laplace Transformation

Definition: $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$.

Useful Formulae:

(1) First derivatives: $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

(2) Second derivatives: $\mathcal{L}\{f''(t)\} = s^2 F(s) - s f(0) - f'(0)$

(3) Exponential multiplier: $\mathcal{L}\left\{e^{at} f(t)\right\} = F(s-a)$

Table of Laplace Transform

$f(t)$	$F(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$\frac{e^{at}-e^{bt}}{a-b}$, for $a \neq b$	$\frac{1}{(s-a)(s-b)}$
$\frac{ae^{at}-be^{bt}}{a-b}$, for $a \neq b$	$\frac{s}{(s-a)(s-b)}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\frac{1}{(b^2-a^2)}\left(\frac{\sin at}{a}-\frac{\sin bt}{b}\right)$, for $a^2 \neq b^2$	$\frac{1}{(s^2+a^2)(s^2+b^2)}$
$\frac{1}{(b^2-a^2)}(\cos at - \cos bt)$, for $a^2 \neq b^2$	$\frac{s}{(s^2+a^2)(s^2+b^2)}$
t^n, n being a positive integer	$\frac{n!}{s^{n+1}}$
$\frac{t \sin \omega t}{2\omega}$	$\frac{s}{(s^2+\omega^2)^2}$
$\frac{\sin \omega t - \omega t \cos \omega t}{2\omega^3}$	$\frac{1}{(s^2+\omega^2)^2}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$t \cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$

Table of Standard Integrals

	$f(x)$	$\int f(x)\,dx$
1	$x^n,\ n \neq -1$	$\dfrac{x^{n+1}}{n+1}$
2	$\dfrac{1}{x}$	$\ln\lvert x\rvert$
3	e^{ax}	$\dfrac{1}{a}e^{ax}$
4	$\sin x$	$-\cos x$
5	$\cos x$	$\sin x$
6	$\tan x$	$-\ln\lvert\cos x\rvert = \ln\lvert\sec x\rvert$
7	$\cot x$	$\ln\lvert\sin x\rvert$
8	$\sec x$	$\ln\lvert\sec x + \tan x\rvert$
9	$\csc x$	$-\ln\lvert\csc x + \cot x\rvert$
10	$\sec x \tan x$	$\sec x$
11	$\sec x \tan x$	$-\csc x$
12	$\dfrac{1}{a^2 + x^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$
13	$\dfrac{1}{a^2 - x^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{a+x}{a-x}\right\rvert,\ \text{for}\ \lvert x\rvert < a$
14	$\dfrac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$
15	$\dfrac{1}{\sqrt{a^2 + x^2}}$	$\ln\left\lvert x + \sqrt{x^2 + a^2}\right\rvert$
16	$\dfrac{1}{\sqrt{x^2 - a^2}}$	$\ln\left\lvert x + \sqrt{x^2 - a^2}\right\rvert$