# THE HONG KONG POLYTECHNIC UNIVERSITY

Department of Applied Mathematics

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Subject Code: AMA201 / AMA203

Subject Title: Mathematics I / Mathematics IA

Programmes:
BEng (Hons) in Electrical Engineering (41070-1)
Higher Diploma in Electrical Engineering (41073-1)
Part-time BEng (Hons) in Electrical Engineering (41076-SF-1)
BEng (Hons) in Electronic and Information Engineering (42070-1)
Higher Diploma in Electronic and Information Engineering (42075-1)
BEng (Hons) in Mechanical Engineering (43078-1)
BEng (Hons) in Mechanical Engineering (43078-2)
Higher Diploma in Mechanical Engineering (43079-1)
BEng (Hons) in Manufacturing Engineering (45074-1)
BEng (Hons) in Manufacturing Engineering (45084-1)

Session: Semester 2, 2000/2001

Date: 18 May 2001  Time: 6:30 – 9:30 p.m.

Time Allowed: 3 Hours

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This question paper has 5 pages (attachments included).

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Instructions to Candidates: This question paper has SIX questions.
Attempt FIVE questions.
All questions carry equal marks.

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Attachment: Laplace Transform Table

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Subject Examiner: Mr. F.Y. Sing

**DO NOT TURN OVER THE PAGE UNTIL YOU ARE TOLD TO DO SO**

(1) (a) Solve the following system of linear equations

$$\begin{cases} 3x_1 & -2x_2 & +x_3 & =-3 \\ x_1 & +3x_2 & +4x_3 & =10 \\ -2x_1 & +5x_2 & +2x_3 & =9 \end{cases}.$$

[6 marks]

(b) Let $\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \end{bmatrix}$. Find the eigenvalues and corresponding eigenvectors of $\mathbf{C}$. Hence evaluate

$$\lim_{n\to\infty} \mathbf{C}^n \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

[9 marks]

(c) Determine if the vectors $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$, $\begin{bmatrix} 0 & 4 & -2 \end{bmatrix}^T$, $\begin{bmatrix} 1 & -1 & -5 \end{bmatrix}^T$ are linearly independent. Justify your answer.

[5 marks]

(2) (a) Find the relation between the scalars $s$ and $t$ such that the matrix

$$\mathbf{M} = \begin{bmatrix} 5 & 3 & 1 \\ 3 & -1 & 2 \\ s & t & -1 \end{bmatrix}$$

is nonsingular. Find $\mathbf{M}^{-1}$ if $s = 1$ and $t = 0$.

[6 marks]

(b) Let $\mathbf{B} = \begin{bmatrix} 8 & 8 & -8 \\ 10 & -3 & 1 \\ 10 & 5 & -7 \end{bmatrix}$. Find the eigenvalues and the corresponding eigenvectors of $\mathbf{B}$. Hence find a nonsingular matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{PDP}^{-1} = \mathbf{B}$.

[14 marks]

(3) (a) By the method of *variation of parameters*, find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = \frac{e^{-2x}}{x^2},\ x > 0.$$

[10 marks]

(b) Using the method of undetermined coefficients, find constants $A$, $B$, $C$ and $D$ so that $y_p(x) = (Ax + B)\cos 2x + (Cx + D)\sin 2x$ is a particular solution of the differential equation

$$\frac{d^2y}{dx^2} + y = 3\sin 2x + x\cos 2x.$$

Hence, obtain the general solution of the differential equation.

[10 marks]

(4) (a) Solve the first order differential equation

$$\frac{dy}{dx} + \left(\frac{2}{x}\right)y = \frac{x}{x^4 + 1}.$$

[8 marks]

(b) Use Laplace Transform to find the solution of the initial value problem

$$\frac{d^2y}{dt^2} + 9y = 3\sin 2t,$$

given that $y(0) = -1$ and $y'(0) = 1$.

[12 marks]

(5) (a) The demand functions for products $A$ and $B$ are defined respectively by

$$q_A = \frac{50\sqrt[3]{y}}{\sqrt{x}} \quad \text{and} \quad q_B = \frac{75x}{\sqrt[3]{y}},$$

where $x > 0$ and $y > 0$ denote the respective prices of products $A$ and $B$. Compute the partial derivatives $\frac{\partial q_A}{\partial x}$, $\frac{\partial q_A}{\partial y}$, $\frac{\partial q_B}{\partial x}$ and $\frac{\partial q_B}{\partial y}$. Hence deduce that the two products are competitive, that is, $\frac{\partial q_A}{\partial y} > 0$ and $\frac{\partial q_B}{\partial x} > 0$.

[6 marks]

(b) Let $w = g(u)$ be a differentiable function of $u$, and suppose $w = g(x^2 - 2xy)$. Use the chain rule of differentiation to show that

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial x \partial y} = 0 \text{ when } x = y.$$

[6 marks]

(c) Find all critical points of the function

$$f(x, y) = \frac{x^3 + 8y^3}{3} - 2\left(x^2 + y^2\right)$$

and determine their nature.

[8 marks]

(6) (a) Use Lagrange Multiplier Method to find the maximum of $f(x, y, z) = xyz$, where $x > 0$, $y > 0$, $z > 0$, subject to the constraint $x + 2y + 3z = 36$.

[7 marks]

(b) The polar and the rectangular coordinates are related by the equations

$$\begin{cases} r\cos\theta - x &= 0 \\ r\sin\theta - y &= 0 \end{cases}.$$

(i) Assuming that $x$ and $y$ are defined as functions of $r$ and $\theta$ by these equations, compute $\dfrac{\partial x}{\partial r}$, $\dfrac{\partial y}{\partial r}$, $\dfrac{\partial x}{\partial \theta}$ and $\dfrac{\partial y}{\partial \theta}$.

(ii) Assuming that $r$ and $\theta$ are defined as functions of $x$ and $y$ by these equations, use implicit differentiation to compute $\dfrac{\partial r}{\partial x}$, $\dfrac{\partial \theta}{\partial x}$, $\dfrac{\partial r}{\partial y}$ and $\dfrac{\partial \theta}{\partial y}$.

(iii) Verify that $\begin{bmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\ \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial y} \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{bmatrix}$

[8 marks]

(c) Use total differential to approximate

$$\sqrt[3]{1.98^2 + 2.03^2}.$$

[5 marks]

# Laplace Transform Table

**Definition:** $F(s) = \int_0^\infty e^{-st} f(t)dt.$

**Useful Formulas:** Suppose $F(s) = \mathfrak{L}\{f(t)\}$. *Then*

(1) First derivative: $\mathfrak{L}\{f'(t)\} = s\,F(s) - f(0);$

(2) Second derivative: $\mathfrak{L}\{f''(t)\} = s^2\,F(s) - s\,f(0) - f'(0);$

(3) Integral: $\mathfrak{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s};$

(4) Exponential multiplier: $\mathfrak{L}\{e^{at}\,f(t)\} = F(s-a).$

# Table of Laplace Transform

| $f(t)$ | $F(s)$ |
|---|---|
| $1$ | $\frac{1}{s}$ |
| $e^{at}$ ($a$ being a real constant) | $\frac{1}{s-a}$ for $s > a$ |
| $\frac{e^{at} - e^{bt}}{a-b}$ for $a \neq b$ | $\frac{1}{(s-a)(s-b)}$ |
| $\frac{a\,e^{at} - b\,e^{bt}}{a-b}$ for $a \neq b$ | $\frac{s}{(s-a)(s-b)}$ |
| $\cos \omega t$ ($\omega > 0$) | $\frac{s}{s^2+\omega^2}$ |
| $\sin \omega t$ ($\omega > 0$) | $\frac{\omega}{s^2+\omega^2}$ |
| $\frac{1}{(b^2-a^2)}\left(\frac{\sin at}{a} - \frac{\sin bt}{b}\right)$ for $a^2 \neq b^2$ | $\frac{1}{(s^2+a^2)(s^2+b^2)}$ |
| $\frac{1}{(b^2-a^2)}(\cos at - \cos bt)$ for $a^2 \neq b^2$ | $\frac{s}{(s^2+a^2)(s^2+b^2)}$ |
| $t^n$ ($n$ being a positive integer) | $\frac{n!}{s^{n+1}}$ |
| $\frac{t \sin \omega t}{2\omega}$ | $\frac{s}{(s^2+\omega^2)^2}$ |
| $\frac{\sin \omega t - \omega t \cos \omega t}{2\omega^3}$ | $\frac{1}{(s^2+\omega^2)^2}$ |
| $e^{-\alpha t} \sin \omega t$ | $\frac{\omega}{(s+\alpha)^2+\omega^2}$ |
| $e^{-\alpha t} \cos \omega t$ | $\frac{s}{(s+\alpha)^2+\omega^2}$ |
| $t \cos \omega t$ | $\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$ |