Unit 1 Indices and logarithms

Learning Objectives

The students should be able to:

- Use the laws of indices to solve simple problems.
- Use the properties of logarithms to solve simple problems.

Indices

In dealing with expressions containing exponents along with addition or subtraction or multiplication or division, we work with the exponents first. Lets consider the following examples.

Example 1 (a) $3^2 - 2^3 = 9 - \_\_\_ = \_\_\_\_$

(b) $2^4 + 5^3 = 16 + \_\_\_ = \_\_\_\_\_$

(c) $3 \text{ x } 10^4 = 3 \text{ x } \_\_\_\_\_\_\_\_ = 30000$

Example 2 Write each of the following in another way by using exponents.

(a) (ab)(ab)(ab)
(b) –a.a.a.a
(c) 4.a.b.4.b.a.a.a

Solution:

(a) $(ab)(ab)(ab) = (ab)^{\_\_\_}$
$= a^{\_\_} b^{\_\_}$
(b) $-a.a.a.a = -a^{\_\_}$
(c) $4.a.b.4.b.a.a.a.b = 4^2 a^{\_\_} b^{\_\_}$

Example 3 Evaluate each of the following :

(a) -3^4
(b) $(-3)^4$
(c) $2(1.1)^3$

Solution:

(a) $-3^4 = -1(3)^4 = -1.\ 3.3.\_\_.\_\_ = -\_\_\_\_\_$
(b) $(-3)^4 = (-3)(-3)(\_\_\_)(\_\_\_) = \_\_\_\_\_$

(c) $2(1.1)^3 = 2 \text{ x } 1.1^3$

$= 2 \text{ x } \_\_\_\_\_\_ = \_\_\_\_\_\_\_\_$

1. Multiplication of Exponential Numbers

Consider

$2^3.2^4 = (2.2.2)(2.2.2.2)$
$= 2.2.2.2.2.2.2$
$= 2^7$

$a^2.a^3 = (a.a)(a.a.a)$
$= a.a.a.a.a$
$= a^5$

In general, $\boxed{a^m.a^n = a^{m+n}}$

2. Division of Exponential Numbers

Consider

$$\frac{4^7}{4^3} = \frac{4.4.4.4.4.4.4}{4.4.4}$$
$$= 4^4$$

$$\frac{a^5}{a^3} = \frac{a.a.a.a.a}{a.a.a}$$
$$= a^2$$

In general ,

$$\boxed{\frac{a^m}{a^n} = a^{m-n}}$$

Example 4 Simplify the following :

a) $\frac{3^{19}}{3^{11}} = 3^{19-11} = \_\_\_$

b) $\frac{X^9}{X^3} = X^{---} = \_\_\_\_\_$

c) $\frac{a^{10}b^8}{a^3b^5} = a^{10-3}b^{--} = a^7b^{--}$

Consider $\frac{a^n}{a^n} = a^{n-n}$

$= a^0$………………..(1)

On the other hand

$\frac{a^n}{a^n} = 1$……….………..…….(2)

Compare (1) & (2): $\boxed{a^0 = 1}$

where a is any real number and $a \neq 0$

For examples,

$$2^0 = 1, \qquad (-3)^0 = 1 \quad \text{and} \qquad 10^0 = 1$$

Recall $\dfrac{a^m}{a^n} = a^{m-n}$ ………………(*)

Consider $\dfrac{2^3}{2^4} = \dfrac{2\cdot 2\cdot 2}{2\cdot 2\cdot 2\cdot 2} = \dfrac{1}{2}$

If we use the division rule (*)

$$\frac{2^3}{2^4} = 2^{3-4} = 2^{-1}$$

$$\therefore \quad 2^{-1} = \frac{1}{2}$$

Consider $\dfrac{3^4}{3^6} = \dfrac{3\cdot 3\cdot 3\cdot 3}{3\cdot 3\cdot 3\cdot 3\cdot 3\cdot 3} = \dfrac{1}{3^2}$

If we use the division rule (*)

$$\frac{3^4}{3^6} = 3^{4-6} = 3^{-2}$$

$$\therefore \quad 3^{-2} = \frac{1}{3^2}$$

In general $a^{-n} = a^{0-n} = \dfrac{a^0}{a^n}$

$$\boxed{a^{-n} = \frac{1}{a^n}}$$

$$\boxed{(a\,b)^n = a^n\, b^n} \qquad \boxed{\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}} \qquad \boxed{(a^m)^n = a^{mn}}$$

Example 5 Simplify the following :

(a) $(a^2 b)^3$

(b) $(\dfrac{x^3}{y})^2 (y^3)^2$

Solution :

(a) $(a^2 b)^3$ = a__ b__

(b) $(\dfrac{x^3}{y})^2 (y^3)^2$ = x__ y__

Example 6 Carry out each of the following operation. Write all answers with positive exponents and simplify where possible.

(a) 7^{-1} (b) 2^{-4}

(c) $5(5^{-3})$ (d) $\dfrac{2^{-5}}{2^{3}}$

Solutions:

(a) $7^{-1} = \dfrac{1}{7}$

(b) $2^{-4} = \dfrac{1}{2^{-}}$ *or* $\dfrac{1}{\_\_\_}$ $=$

(c) $5(5^{-3}) = 5^{1} \times 5^{-3}$

$= 5^{1+(-3)} = 5^{-2}$

$= \dfrac{1}{\_\_\_}$ *or* $\dfrac{1}{\_\_\_}$

(d) $\dfrac{2^{-5}}{2^{3}} = 2^{-5-3} = 2^{-8}$

$= \dfrac{1}{\_\_\_}$ *or* $\dfrac{1}{256}$

3. Radicals

If $x^2 = y$, then x is a square root of y

for example $7^2 = 49$

$\therefore$ $7 = \sqrt[2]{49}$ or $7 = \sqrt{49}$

similarly, $\sqrt{81} = 9$ ($\because$ $9^2 = 81$)

If $x^3 = y$, then x is a cube root of y

for example, $4^3 = 64$ ($\because$ $\sqrt[3]{64} = 4$)

In general, if $x^n = y$, where n is a positive integer, then x is a n^{th} root of y.
for example, $2^4 = 16$

$\therefore$ $\sqrt[4]{16} = 2$

Example7 Find the values of the following :

(a) $\sqrt[6]{64}$

(b) $\sqrt[5]{100000}$

(c) $\sqrt[4]{81}$

Solution : (a) $\sqrt[6]{64} = \_\_$

($\because \; 2^6 = 64$)

(b) $\sqrt[5]{100000} = \_\_\_$

($\because \; 10^5 = 100000$)

(c) $\sqrt[4]{81} = \_\_\_$

($\because$ 3−−=81)

Consider $144 = 9 \cdot 16$

and $\sqrt{144} = 12$

$\sqrt{9} \cdot \sqrt{16} = 3 \cdot 4$

$= 12$

$\therefore \; \sqrt{144} = \sqrt{9}\sqrt{16}$

$\sqrt{9 \cdot 16} = 3 \cdot 4$

In general $\boxed{\sqrt{ab} = \sqrt{a}\sqrt{b}}$ $\boxed{\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}}$

Note that $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$

e.g. $\sqrt{25} = \sqrt{16+9}$

but $\sqrt{16} + \sqrt{9} = 4 + 3 = 7$

$\sqrt{25} = 5$

$\therefore \; \sqrt{16+9} \neq \sqrt{16} + \sqrt{9}$

In general $\boxed{\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}}$ $\boxed{\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}}$

Example 8 Find the values of the following functions :

(a) $f(x) = \sqrt{\frac{x}{64}}$ when $x = 25$

(b) $f(\text{x}) = x^{0.5}$ when $x = 0.027$

Solution : (a) $f(25) = \sqrt{\frac{25}{\_\_\_}} = \frac{\sqrt{25}}{\sqrt{\_\_\_}}$

$= \frac{5}{\_\_\_}$

(b) $f(0.027) = \sqrt[3]{\frac{27}{\_\_\_}} = \frac{\sqrt[3]{27}}{\sqrt[3]{\_\_\_}}$

$= \frac{3}{\_\_\_}$ or 0.3

4. Fractional Indices

Consider $\left(a^{\frac{1}{n}}\right)^n = a^{\frac{1}{n}\cdot n}$

$= a^1 = a$

$$\therefore\ a^{\frac{1}{n}} = \sqrt[n]{a}$$

Similarly , $a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$

Example 9 Evaluate the following:

(a) $81^{\frac{3}{4}}$ (b) $1000^{\frac{2}{3}}$ (c) $\left(\frac{9}{25}\right)^{\frac{1}{2}}$

Solutions:

(a) $81^{\frac{3}{4}} = (\sqrt[4]{\_\_\_})^3 = (\_\_\_)^3 = \_\_\_$ (b) $1000^{\frac{2}{3}} = (\sqrt[3]{\_\_\_})^2 = (10)^2 = \_\_\_$

(c) $\left(\frac{9}{25}\right)^{\frac{1}{2}} = \sqrt{\frac{9}{\_\_\_}} = \frac{\sqrt{9}}{\sqrt{\_\_\_}} = \frac{3}{\_\_\_}$

Example 10 In the compound interest formula $A = P(1+r)^n$, if $A = 54874.32$, $P = 25000$, $n = 6$, find the value of r.

Solution: $A = P(1+r)^n$

$54874.32 = (25000)(1+r)^6$

$\frac{54874.32}{25000} = (1+r)^6$

$\therefore 1+r = \sqrt[6]{\frac{54874.32}{25000}}$

$1 + r =$ _____

$r =$ _____ $- 1$

$= 0.14$

$=$ ___%

5. Definition of Logarithms

If a number $X = a^y$, where **a** is positive and **a? 1**, the index **y** is called the **logarithm** of the number X to the base **a**. In symbol, **y** = $\log_a X$.

N.B. $\log_a X$ is undefined only for positive values of X

For example, $2^3 = 8$

$\therefore \log_2 8 = 3$

$3^4 = 81$

$\therefore \log_3 81 = 4$

When the base a is not stated in $\log_a X$, it may be assumed $a = 10$. This is called the **common logarithm.**

For example, $\log 1000 = \log_{10} 1000$

$= 3$

$\log 0.01 = -2$

$(\because 10^{-2} = \left(\frac{1}{10^2}\right) =$ ____)

6. Properties of Logarithms :

1. $\log_a a = 1$
2. $\log_a 1 = 0$
3. $\log_a MN = \log_a M + \log_a N$
4. $\log_a \frac{M}{N} = \log_a M - \log_a N$
5. $\log_a X^n = n \log_a X$

Example 11 Find the values of the following:

(a) $\log_7 11 + \text{Log}_7\left(\frac{1}{11}\right)$

(b) $\log 6 - \log 60$

(c) $\log_5 125$

Solution:

(a) $\log_7 11 + \log_7\left(\frac{1}{11}\right)$

$\log_7\left(11 \times \frac{1}{\_\_\_}\right) = \log_7 1 = \_\_\_$

(b) $\log 6 - \log 60$

$\log \frac{6}{\_\_\_} = \log \frac{1}{10}$

$\log 10^{-1} = \_\_\_\_$

(c) $\log_5 125$

$= \log_5 5^3$

$= 3 \log_5 \_\_\_$

$= \_\_\_\_$

Example 12 In the compound interest formula $A = P(1+r)^n$, if $A = 20000$, $P = 10000$,

$r = 12\%$, find n correct to 2 decimal places.

Solutions: $A = P(1+r)^n$

$20000 = 10000(1 + 12\%)^{-}$

$$\frac{20000}{10000} = (1+0.12)^n$$

$2 = (1.12)^n$

Taking logarithm on both sides, $\log 2 = \log(1.12)^n$

$\log 2 = n \log 1.12$

$$\therefore n = \frac{\log 2}{\log 1.12}$$

n = ____

= ____