Unit 10: Trigonometric ratios and their graphs

Learning Objectives

Students should be able to

- define positive angles and negative angles
- define the measurement of an angle in radians
- define the trigonometric ratios of angles between 0^o and 360^o (equivalently 0 to 2π radians)
- evaluate trigonometric ratios of angles between 0^o and 360^o by calculators (equivalently 0 to 2π radians)
- plot the graphs of simple trigonometric ratios between 0^o and 360^o (equivalently 0 to 2π radians)
- apply trigonometric graphs to solve simple daily problems

Trigonometric ratios and their graphs

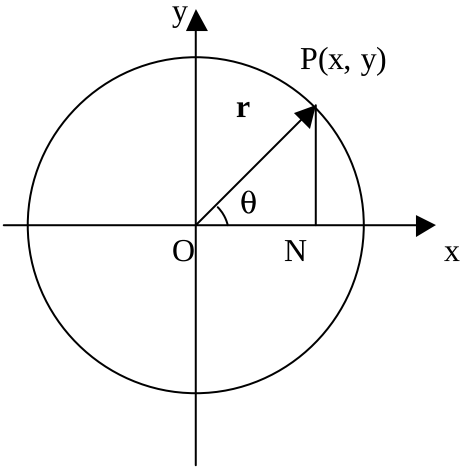
1. Angles of Rotation

The concept of angles of rotation enables us to define and evaluate the trigonometric ratios for angles greater than 90^o.

1.1 Positive and negative angles

In figure 1, a unit vector **r** is rotating in the anti-clockwise direction about a fixed point O and a positive angle θ is formed. When **r** is rotating in the clockwise direction, θ would be negative.

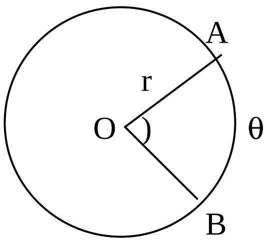
Figure 1



At time t, angle xOP = θ.

1.2 Circular Measurement

Figure 1



In Figure 1, L = length of arc AB

θ = angle at the centre of sector measured in radians (circular measure)

r = radius of the circle

From proportionality, we have

length of arc: circumference of circle = angle at the centre : angle of a whole circle

i.e. $L : 2\pi r = \theta : 2\pi$

$$\boxed{L = r\theta}$$

In particular, if the arc length of the sector is equal to its radius, then the angle is 1 **radian**. It is obvious that the size of an angle of 360^o is equivalent to 2π radians. Therefore, 1 radian is approximately equal to $(180/\pi)^o = 57.3^o$.

> Converting angles from degrees to radians would be done by multiplying the factor (π rad/180^o).
> Converting angles from radians to degrees would be done by multiplying the factor ($180^o/\pi$ rad).

Note: The size of a radian does not depend on the size of a circle.

<u>Example 1</u>

Calculate the following angles in degrees:

a. 1.3 rad b. 1.5π rad

<u>Solution</u>

a. 1.3 rad = $1.3 \times 180^o/\pi = 74.5^o$

b. 1.5πrad = ( )πx( )/π = ( )^o

<u>Example 2</u>

Express the following angles in radians:

a. 18^o b. 178^o

<u>Solution</u>

a. $18^o = 18 \times \pi$ rad/180 = 0.314 rad

b. $178^o = 178 \times \pi$ rad/( ) = ( )rad

<u>Example 3</u> Calculate the arc length of a sector of radius 10 cm and sector angle 18^o.

<u>Solution</u> $L = r\theta = 10\left(\frac{\pi}{180}\right) = \pi$ cm (cm)

The following table shows the conversion of some special angles:

Angle in degrees	0^o	30^o	60^o	90^o	180^o	270^o	360^o
Angle in radians	0	$\pi/6$	$\pi/3$	$\pi/2$	π	$3\pi/2$	2π

2.1 Trigonometric ratios for angles between 0^o and 90^o (0 to $\frac{p}{2}$ rad)

For $\theta < 90^o$, we have

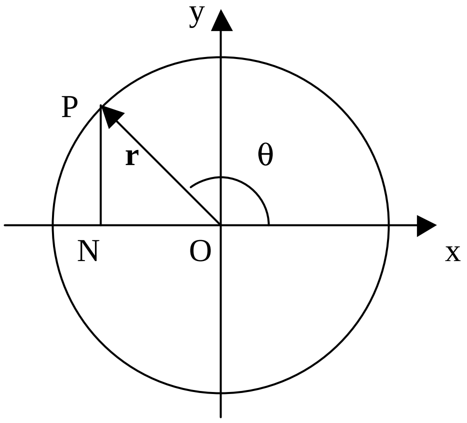
$$\sin\theta = PN/r \quad , \quad \cos\theta = ON/r \quad , \quad \tan\theta = PN/ON$$
$$= y/r \qquad\qquad = x/r \qquad\qquad = y/x$$

Please note that all the ratios sine, cosine and tangent are positive in this case.

2.2 Trigonometric ratios for angles between 90^o and 180^o ($\frac{p}{2}$ to p rad)

In Figure 2, $90^o < \theta < 180^o$, we define the trigonometric ratios as follows:

Figure 2



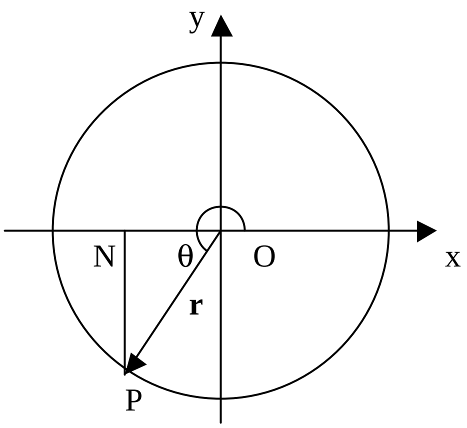
$$\sin\theta = y/r \quad , \quad \cos\theta = x/r \quad , \quad \tan\theta = y/x$$

where x is the x-coordinate of P and y is the y-coordinate of P.
Please note x is negative in this case. Subsequently, the ratio of sine is positive while the ratios of cosine and tangent are negative.

2.3 Trigonometric ratios for angles between 180° and 270° (π to $\frac{3\pi}{2}$ rad)

In Figure 3, $180^{o} < \theta < 270^{o}$, the trigonometric ratios are defined as follows:

Figure 3



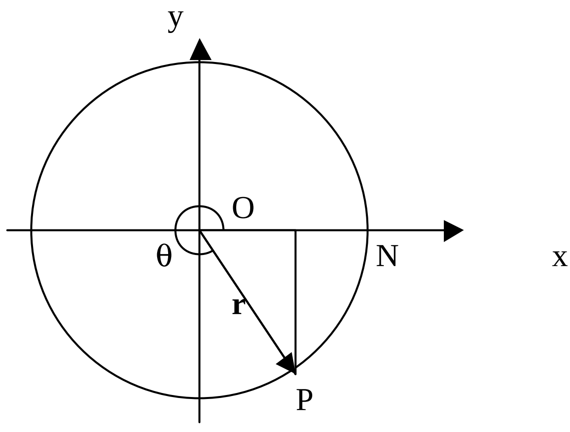
$$\sin\theta = y / r \quad , \quad \cos\theta = x / r \quad , \quad \tan\theta = y / x$$

Please note both x, y are negative in this case. Subsequently, the ratio of tangent is positive while the ratios of sine and cosine are negative.

2.4 Trigonometric ratios for angles between 270° and 360° ($\frac{3\pi}{2}$ to 2π rad)

In Figure 4, $270^{o} < \theta < 360^{o}$, we define the trigonometric ratios as follows:

Figure 4



$$\sin\theta = y / r \quad , \quad \cos\theta = x / r \quad , \quad \tan\theta = y / x$$

Please note that y is negative in this case. Subsequently, the ratio of cosine is positive while the ratios of sine and tangent are negative.

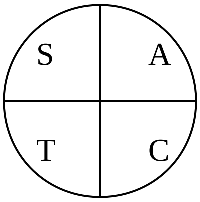
In summary, the definition of the trigonometric ratios are as follows:

$\sin\theta$ = y-projection/ r, $\cos\theta$ = x-projection/ r

$\tan\theta$ = y-projection/ x-projection

3. The CAST Rule

The signs of the trigonometric ratios can easily be memorized by writing the word CAST in the quadrants.



Summary

In the first quadrant,	All ratios are positive.
In the second quadrant,	Sine is positive.
In the third quadrant,	Tangent is positive.
In the fourth quadrant,	Cosine is positive.

3.1 Numerical values of trigonometric ratios

Numerical values of trigonometric ratios can easily be found by using calculators.

<u>Example 4</u> By using calculators, show that the values tabulated below are correct.

θ	12^o	100^o	207^o	302^o	-12^o	1.2 rad
$\sin\theta$	0.2079	( )	-0.4540	( )	-0.2079	( )
$\cos\theta$	( )	-0.1736	( )	0.5299	( )	0.3624
$\tan\theta$	0.2126	( )	0.5095	-1.6003	-0.2126	( )

4. Graphs of trigonometric ratios

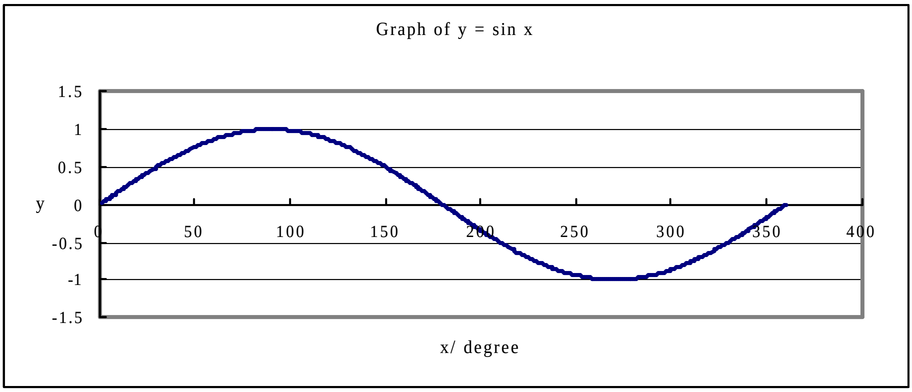
The graphs of trigonometric ratios have very practical applications in many daily situations in economic and engineering regimes. With the use of calculators, the values of a trigonometric ratio can readily be tabulated.

4.1 The sine graph

First of all, we have to write down the values of the ordered pairs x and y in a table.
Here x represent the angle in degrees while y = sin x.

x	0^o	30^o	60^o	90^o	120^o	150^o	180^o	210^o	240^o	270^o	300^o	330^o	360^o
y	0	0.5	0.87	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0

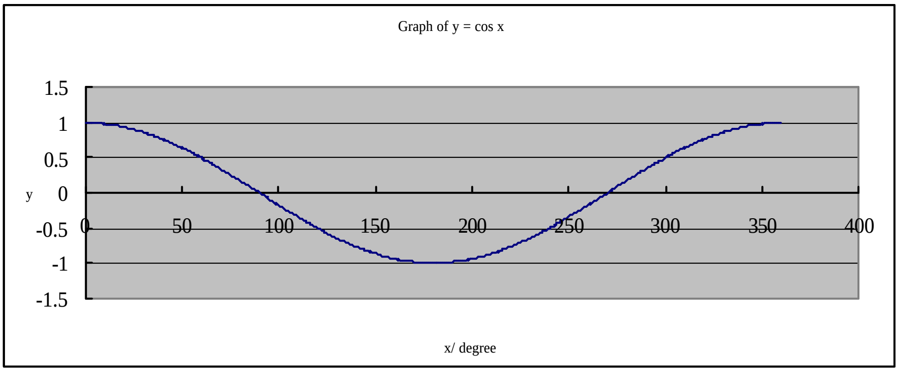
By careful drawing, a smooth sine graph is formed.



4.2 The cosine graph

By writing down the values of the ordered pairs x and y in a table, a cosine graph is formed.
Here x represent the angle in degrees while y = cos x.

x	0^o	30^o	60^o	90^o	120^o	150^o	180^o	210^o	240^o	270^o	300^o	330^o	360^o
y	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0	0.5	0.87	1

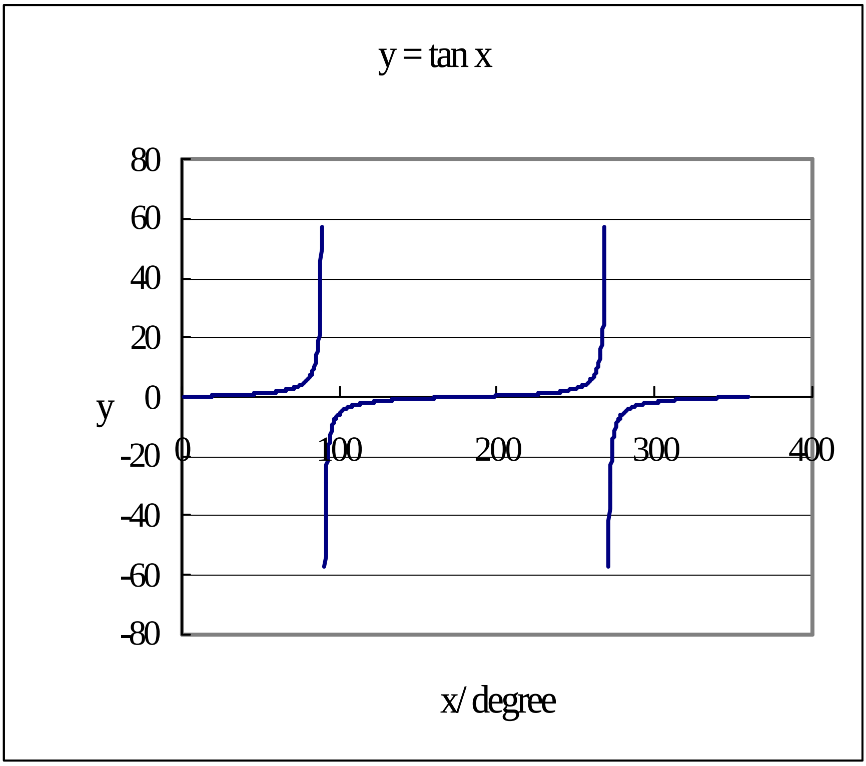


Note: Both sine and cosine graphs are called sinusoidal curves.

4.3 The tangent graph

By writing down the values of the ordered pairs x and y in a table, a tangent graph can similarly be formed. Here x represent the angle in degrees while y = tan x.

x	0^o	30^o	60^o	90^o	120^o	150^o	180^o	210^o	240^o	270^o	300^o	330^o	360^o
y	0	0.58	1.73	8	-1.73	-0.58	0	0.58	1.73	8	-1.73	-0.58	0



Note: The graph of the tangent function is not a continuous curve.

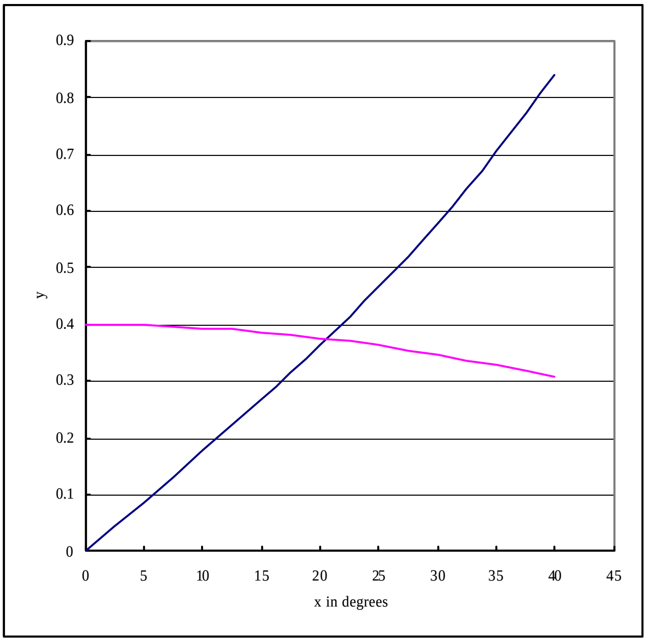
Example 5

Solve the equation 5 $\tan x = 2\cos x$ graphically for $0 < x < \frac{\pi}{2}$.

Solution: The equation reduces to $\tan x = 0.4\cos x$

By plotting the graphs of $y = \tan x$ and $y = 0.4\cos x$,

x/deg	0^o	10^o	20^o	30^o	40^o	50^o	60^o	70^o	80^o
x/rad	0	$\frac{\pi}{18}$	$\frac{\pi}{9}$	$\frac{\pi}{6}$	$\frac{2\pi}{9}$	$\frac{5\pi}{18}$	$\frac{\pi}{3}$	$\frac{7\pi}{18}$	$\frac{4\pi}{9}$
$\tan x$	0	0.176	0.364	0.577	0.839	1.19	1.73	2.75	5.67
$0.4\cos x$	0.4	0.39	0.38	0.35	0.31	0.26	0.20	0.14	0.07



the intersection of the two curves gives x = (   ) rad(^{o}).

Web Fun

Try the Polar bearing game at http://www.ex.ac.uk/cimt/