Unit 12: The Sine formula and the Cosine formula

Objectives

Students should be able to

- state the sine formula
- state the cosine formula
- apply the sine formula to solve simple problems
- apply the cosine formula to solve simple problems

The Sine formula and the Cosine formula

1.1 The Sine formula

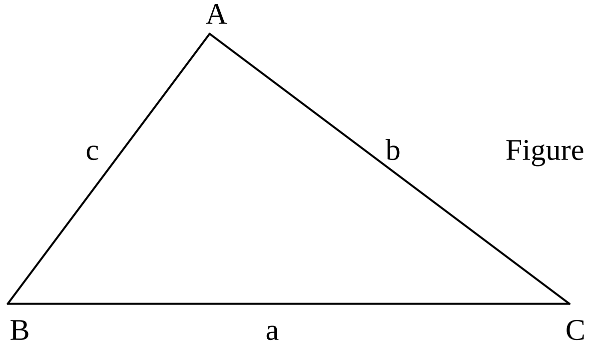


Figure 1

In any triangle ABC (figure 1),

The area of the triangle can be expressed as $\frac{1}{2}ab\sin C$ or $\frac{1}{2}bc\sin A$ or $\frac{1}{2}ac\sin B$. Hence $\frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B$.

Dividing throughout by $\frac{1}{2}abc$, we obtain the Sine formula :

$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}}$$

i.e. $\frac{a}{\sin A} = \frac{b}{\sin B}$ or $\frac{b}{\sin B} = \frac{c}{\sin C}$ or $\frac{c}{\sin C} = \frac{a}{\sin A}$

1.2 The Cosine formula

The cosine formula can be established for any triangles. We consider the cases where C is acute and C is obtuse.

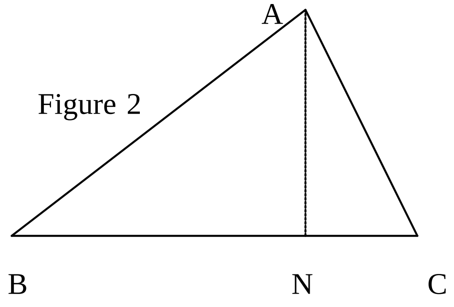


Figure 2

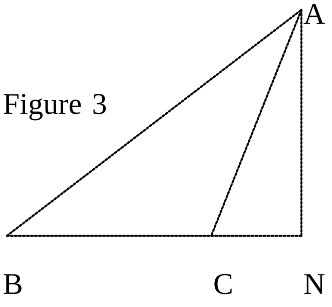


Figure 3

In figure 2, $AB^2 - BN^2 = AN^2 = AC^2 - CN^2$

Leading to $c^2 - (a-x)^2 = AN^2 = b^2 - x^2$ where $CN = x$

Which can be simplified to $2ab\cos C = a^2 + b^2 - c^2$ with the substitution $x = b\cos C$

In figure 3, $c^2 - (a+x)^2 = AN^2 = b^2 - x^2$ where $CN = x$

Which can be simplified to $2ab\cos C = a^2 + b^2 - c^2$ with the substitution $x = b\cos(180^o - C)$

Thus, we establish the Cosine formula :

$$c^2 = a^2 + b^2 - 2ab\cos C$$

or

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

We can use these formulas to solve problems on triangles.

2 Application of the Sine Formula

The sine formula can be applied to solve a triangle (I) when two angles and one side or (II) when two sides and a non-included angle of the triangle are given. Care should be taken since ambiguity case may arise in (II).

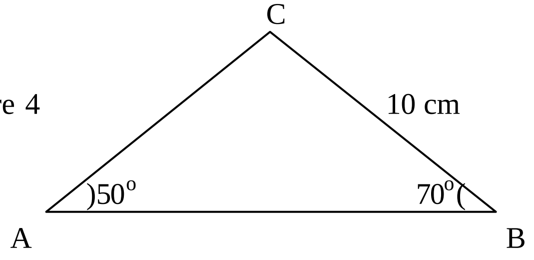
2.1 Two angles and any one side of a triangle are given

<u>Example 1</u>

In ΔABC of figure 4, $A = 50°$, $B = 70°$ and $a = 10$cm. Solve the triangle.

(Answers correct to 3 significant figures if necessary.)

Figure 4



<u>Solution</u>

From subtraction, C = 60^o

By sine formula, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

AC = 10 x sin ___^{o}/ sin 50 ^{o}

= cm

AB = 10 x sin ___^{o}/ sin 50 ^{o}

= cm

2.2 Two sides and one angle of a triangle are given

2.2.1 Two sides and one opposite angle are given

<u>Example 2</u>

In ΔABC, find B if A = 30°, b = 10cm and a = 4cm.

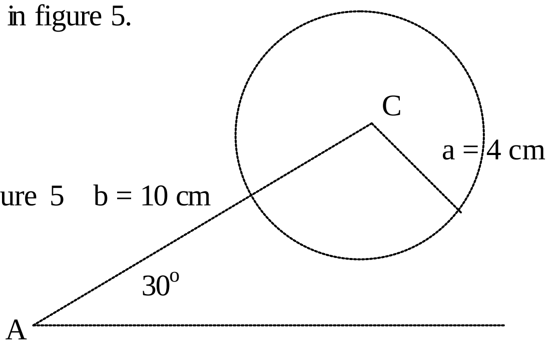
<u>Solution</u>

From sine formula, sinB = 10x sin___^{o}/4

sinB = _____ > 1 which is impossible

Hence, no triangle exists for the data given. The situation can further be illustrated by accurate drawing as in figure 5.

Figure 5 b = 10 cm



<u>Example 3</u>

In ΔABC, find B if A = 30°, b = 10cm and a = 5cm.

<u>Solution</u>

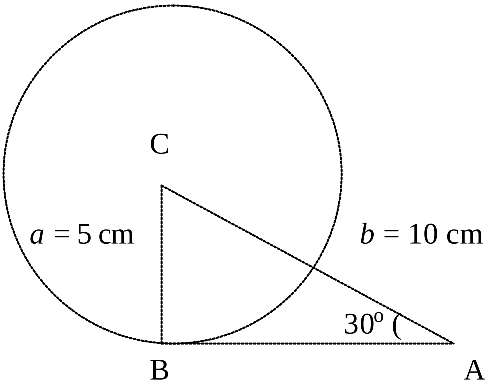
By sine formula, sinB = _____x sin30^o/5

sinB =

B = _____°

In this case, only one right-angled triangle can be drawn as in figure 6.

Figure 6



Example 4

In ΔABC, find B if A = 30°, *b* = 10cm and *a* = 6cm.

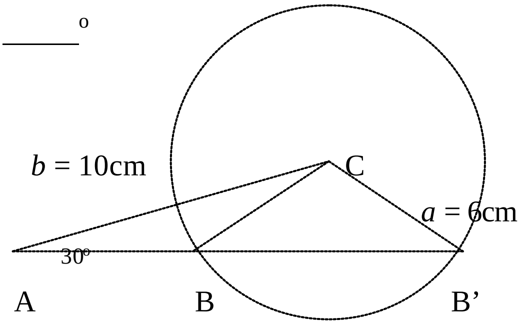
Solution

By sine formula, sinB = 10x sin___ °/6

sinB =

B = ___° , _____°

Figure 7



Two solutions exist as shown in figure 7.

Example 5

In ΔABC, find B if A = 30°, *b* = 10cm and *a* = 15cm.

Solution

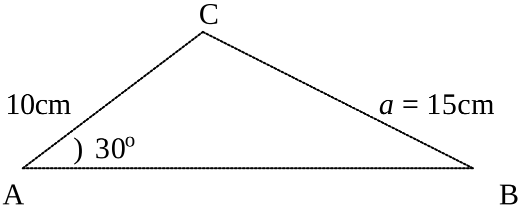
By sine formula, sinB = 10x sin___ °/15

sinB =

B = ____°

Only one solution exists as shown in figure 8. No ambiguity in this case.

Figure 8



From the above examples, we can see that if we apply the sine formula to solve a triangle when 2 sides and 1 opposite acute angle are given, we may have

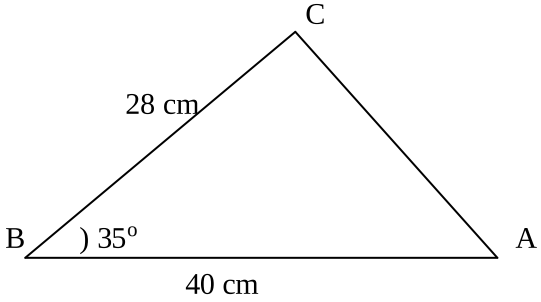
(i) *no solutions* [example 2],
(ii) *one (unique) solution* [example 3 and example 5],
(iii) *two solutions* [example 4].

2.2.2 Two sides and one included angle are given

Example 6

Given a triangle ABC, in which a = 28cm, c = 40cm, B = 35°. Find the length AC and correct your answer to 1 decimal place.

Figure 9



Solution

By cosine formula,

$$b^2 = c^2 + a^2 - 2ca\cos B$$

$b^2 = 28^2 + \_\_\_^2 - 2(28)(\_\_\_\_)\cos 35^\circ$

b = _____ cm

2.2.3 Three sides are given

Example 7

Given a triangle ABC, in which a = 5cm, b = 6cm, c = 7cm. Find the angles of the triangle. (*Correct the answers to the nearest tenth degree.*)

Solution

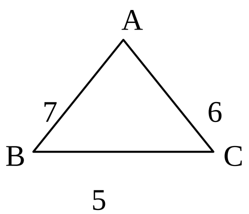


Figure 10

By cosine formula,

$c^2 = a^2 + b^2 - 2ab\cos C$

$7^2 = 5^2 + \_\_\_^2 - 2(\_\_\_\_)(6)\cos C$

cosC =

C= ____^{o}

By sine formula,

sinB = 6 x sin ___^{o}/7

B ____-^{o}

By subtraction, A = ____ ^{o}

2.2.4 Three angles are given

Since the angles do not restrict the size of a triangle, there exists infinitely many triangles that satisfy the given condition.

Example 8

Given a triangle ABC, in which A = 50°, B = 60°, C = 70°. Find the lengths of the sides of the triangle.

Solution

Since the angles do not restrict the size of a triangle, there exists infinitely many triangles that satisfy the given condition. These triangles are all similar.