Unit 18 : Arithmetic Progression (Sequence)

Learning Objectives

Students should be able to

- state the characteristics of progressions
- find the general term of a given progression
- state the characteristics of an arithmetic progression and in particular the common difference
- find the general term of an arithmetic progression with given information
- define arithmetic means
- insert any number of arithmetic means between two given numbers
- state the properties of series and arithmetic series
- apply the two summation formulae to find the sum, number of terms or some particular terms of an arithmetic series

1. Basic terminology of a progression

1.1. Definition and notations

- A **progression** is a collection of numbers arranged in a certain order.

Example 1	The following are examples of progressions	
	1, 2, 3, 4, 5, 6, …	(S1)
	1, 2, 4, 8, 16, 32, …	(S2)
	5, 11, 29, 83, 245, …	(S3)
	2, 3, 5, 7, 11, 13, 17, 19, 23, 29, …	(S4)
	3, 1, 4, 1, 6, 9, 2, 6, 5, 3, …	(S5)

- $T(n)$ represents the n^{th} term.
  For example, $T(1)$ represents the 1^{st} term, $T(2)$ represents the 2^{nd} term…,
  $T(k)$ represents the k^{th} term.

Example 2	Refer to example 1
	$T(1)$ of (S1) is <u>1</u>
	$T(2)$ of (S2) is __
	$T(4)$ of (S3) is __
	$T(6)$ of (S4) is __
	$T(8)$ of (S5) is __

- The **general term** of a progression is a formula which enables us to find any term of the progression. $T(n)$ is commonly used as the symbol for the general term.

Example 3	Write down the third term of the progressions defined by a) $T(n) = 5n + 6$ b) $T(n) = \dfrac{n}{n+2}$		Explanation
	a)	$T(n) = 5n + 6$ $T(3) = 5(\_\_) + \_\_ = \_\_\_\_\_\_$ #	
	b)	$T(n) = \dfrac{n}{n+2}$ $T(3) = \dfrac{\_\_}{\_\_ + 2} = \dfrac{\_\_}{\_\_\_\_}$ #	

2. Arithmetic Progression

2.1. Definition and Notations:

- An **arithmetic progression** is a progression in which *any term minus its previous terms is a constant*. i.e. $T(2) - T(1) = T(3) - T(2) = T(4) - T(3) = \ldots =$ constant. The definition can be written as
  $T(n+1) - T(n) = \text{constant}$ (F1)
- *The constant* is called the **common difference**, *d*.
  $d = T(n+1) - T(n)$ (F2)
- *a* is used to represent the *first term*.

Example 4	Determine which of the following progressions are arithmetic progressions and find their common differences.		Explanation
	a) 2, 7, 12, 17, 22, 27, …	(Y) $d = 5$.	
	b) 1, 5, 10, 14, 20, …	(N)	$T(3)-T(2)\neq T(2)-T(1)$
	c) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$	(N)	
	d) 4, 7, 10, 13, 17	(N)	$T(5)-T(4)\neq T(2)-T(1)$

2.2. The general term

2.2.1. The form of the general term.

Using the symbols *a* and *d*, an arithmetic progression always have the following form:

1st term	2nd term	3rd term	5th term	10th term	nth term	$(m+n)$th term
a	$a + d$	$a + 2d$	$a + 4d$	$a + 9d$	$a + (n-1)d$	$a + (m+n-1)d$

The general term of any AP can be written as

$T(n) = a + (n-1)d$ (F3)

2.2.2. Simple applications of the general term

Applications: to find	Condition
i. The value of a term $T(n)$	The position of the term, n given
ii. The position of a term n	The value of the term $T(n)$ given
iii. The number of terms in the progression	

2.2.3. How to find the general term and examples of applications.

Logical steps	Steps
(F3) is used to find the general term of an AP.	
Thus two unknowns *a* and *d* are to be found and hence two conditions are required	Find the conditions in the question.
Use the conditions to find *a* and *d*	express the conditions into equations, solve the equations
Use (F3)	Subs. *a* and *d* into F3 and simplify the resulting expression.

Example 5	Given that $T(7) = 4$ and $T(20) = -35$, find a) the general term $T(n)$ b) the ninth term of the progression c) m if $T(m) = -86$		Explanation App. i App. ii
solution	a)	$T(n) = a + (n-1)d$ (1) $\_\_ = a + \_\_d$ (2) $-35 = a + \_\_\_\_d$ (3) (2)-(3) 39=-____*d* *d*=____ subs. d into (2) 4 = *a* +6(____) *a* = ____ by (1): $T(n) = \_\_\_\_ + (n-1)(\_\_\_\_)$ $= \_\_\_\_ - 3\_\_$ #	
	b)	*T*(9) = ____-3(__) = ______#	
	c)	$\_\_ - 3m = -\_\_\_\_$ $\_\_\_\_ + \_\_\_\_ = 3m$ $m = \_\_\_\_$ #	

Example 6	Given an arithmetic progression 50, 46, 42, .., -62 , find a) the number of terms that is in the progression. b) the smallest positive term.		Explanation App. iii Other app.
Solution	a)	*a* = ____ *d* = ____ – 50 = ____ $T(n) = a + (n-1)d$ $-62 = 50 - \_\_\_\_(n-1)$ $n = \frac{-62 - \_\_\_\_}{-\_\_\_\_} + 1$ $= \_\_\_\_$ #	
	b)	$50 - \_\_\_\_(n-1) > 0$ $50 + \_\_\_\_ > \_\_n$ $\_\_\_\_ > n$ *n* = ____ is the smallest positive term. Value = 50-4(____-1)= ____#	

3. Arithmetic means

Definition:

The intermediate terms between two terms of an arithmetic progression are called **arithmetic means** between the two terms.

Example 7

Progression	Between	Arithmetic means
2, 3, 4, 5, 6, …	2, 6	3, 4, 5
2, 5, 8, 11, 14, …	2, 11	____, ____

Insert n arithmetic means between a and b. a becomes the 1^{st} term and b the $(n+2)^{th}$ term.

Example 8	Insert 3 arithmetic means between 10 and 20.	Explanation
Solution	$20 = 10 + (\_\_\_\_ -1)d$ $\_\_\_\_ d = 10$ $d = \_\_\_\_$ The means are 12.5, ____, ____#	20 is the 5^{th} term.

The arithmetic mean of two numbers can be found by the same method. However, a simpler formula exists. Clearly, a, "a number at the middle of a and b", and b form an arithmetic progression. By definition the number is the arithmetic mean of a and b.

$$\text{arithmetic mean} = \frac{a+b}{2} \qquad \text{(F4)}$$

Example 9	Find the arithmetic mean of 9 and –1.	Explanation
Solution	$\text{arithmetic mean} = \dfrac{\_\_\_\_ + (-1)}{\_\_\_\_}$ $= \_\_\_\_$ #	

4. Arithmetic Series

A series is a sum of term. 2+4+6+8+10+12+14 is a series, 3+7+5+9 is a series.
We use the $S(n)$ to represent the sum of n term: $S(n) = T(1) + T(2) + T(3) + \ldots + T(n)$.
For an arithmetic progression,

$$S(n) = \frac{n}{2}\left[2a + (n-1)d\right] \qquad \text{(F5)}$$

if we use ℓ to represent the last term, $T(n)$,

$$S(n) = \frac{n}{2}(a+\ell) \qquad \text{(F6)}$$

both of the formulae are useful.

Example 10	Find the sum of the arithmetic series a) 100 + 102 + 104 + …to 20 terms b) 2 + 6 + 10 + 14 + …+ 34		Explanation
Solution	a)	$S(20) = \frac{n}{2}[2a + (n-1)d]$ $= \frac{20}{2}[2(\_\_\_\_) + (\_\_\_\_ - 1)\_\_\_\_]$ $= \_\_\_\_\_\_$ #	(F5) can be used directly.

	b)	$S(n) = \frac{n}{2}(a + \ell)$ (1) $34 = \_\_\_\_ + (n-1)(\_\_\_\_)$ $34 - \_\_\_\_ + \_\_\_\_ = \_\_\_\_n$ $n = \_\_\_\_$ By (1): $S(17) = \frac{\_\_\_\_}{2}(\_\_ + \_\_\_\_)$ $= \_\_\_\__{\#}$	Need to find n first. App. iii

Example 11	$2 + 5 + 8 + 11 + \ldots$, is an arithmetic series. Find m if the sum of the first m terms of the series is 187	Explanation
Solution	$S(m) = \frac{m}{2}\left[2a + (m-1)d\right]$ $\_\_\_\_ = \frac{m}{2}[2(\_\_) + (m-1)(\_\_\_\_)]$ $\_\_\_\_(2) = m(3m-1)$ $\_\_\_\_m^2 + m - \_\_\_\_ = 0$ $m = \frac{-1 \pm \sqrt{1^2 - 4(\_\_\_\_)(-\_\_\_\_)}}{2(\_\_\_\_)}$ $= \_\_\_\__{\#}$ ($m < 0$ is rejected)	Formula substitution