Unit 16 : Geometric Progression (Sequence) and Mortgages

Learning Objectives

Students should be able to

- state the characteristics a geometric progression and in particular the common ratio
- find the general term of a geometric progression with given information
- define geometric means
- insert any number of geometric means between two given numbers
- state the properties of geometric series
- apply the summation formula to find the sum, number of terms or some particular terms of a geometric series
- explain the meaning of the sum to infinity of a geometric series
- find the sum to infinity of a geometric series when $-1 < R < 1$
- Solve mortgage related problems

1 Geometric Progression

1.1 Definition and notations:

A **geometric progression** is a progression in which the ratio of each term to the preceding term is a constant. i.e. $\frac{T(2)}{T(1)} = \frac{T(3)}{T(2)} = \frac{T(4)}{T(3)} = \ldots = \text{constant}$. It can be written as

$$\frac{T(n+1)}{T(n)} = \text{constant} \qquad \text{(F1)}$$

- The constant is called the **common ratio**, *R*.

$$R = \frac{T(n+1)}{T(n)} \qquad \text{(F2)}$$

- *a* is used to represent the first term.

Example 1 Determine which of the following progressions are geometric progressions and find their common ratios. Explanation

a) 2, 4, 6, 8, 10, … (N) $R =$
b) 4, 8, 16, 32, 64, 128, … (Y) $R =$__
c) 25, 5, 1, $\frac{1}{5}$, $\frac{1}{25}$ (Y) $R = \frac{1}{\ }$
d) 4, -12, 36, -108, 354, … (Y) $R =$__

1.2 The general term

Using the symbols *a* and *R*, a geometric progression always has the following form:

1st term	2nd term	3rd term	5th term	10th term	nth term
a	aR	aR^2	aR^4	aR^9	aR^{n-1}

The general term of any geometric progression can be written as

$$T(n) = aR^{n-1} \qquad \text{(F3)}$$

1.1.1. Simple applications of the general term

Application: to find	Condition
i. The value of a term $T(n)$ ii. The position of a term n iii. The number of terms in the progression	The position of the term, n given The value of the term $T(n)$ given

1.1.1. How to find the general term and examples of applications.

Logical steps	Steps
(F3) is used to find the general term of a geometric progression.	
Thus two unknowns *a* and *R* are to be found and hence two conditions are required	Find the conditions in the question.
Use the conditions to find *a* and *R*	express the conditions into equations, solve the equations
Use (F3)	Subs. *a* and *R* into F3 and simplify the resulting expression.

Example 2	A geometric progression is given as follows 5, 20, 80, .., find a) The general term T(n) b) The seventh term c) m if $T(m) = 1280$		Explanation App. i App. ii
solution	a)	$R = \dfrac{T(2)}{T(1)} = \dfrac{\_\_}{5} = \_\_$ $T(n) = aR^{n-1}$ $= \_\_(\_\_)^{n-1}{}_{\#}$	
	b)	$T(7) = 5(\_\_)^{\_\_-1}$ $= \_\_\_\_{}_{\#}$	
	c)	$5(4)^{n-1} = 1280$ $(4)^{n-1} = 256$ $n-1 = \dfrac{\log 256}{\log 4}$ $n = 5_{\#}$	

Although the procedure of finding the general term of a geometric progression is exactly the same as that of an arithmetic progression, there is a major difference in the actual calculation:

- Given any two terms of an AP, only one general term may be found, but given any two terms of a GP, two general terms may be found.

The following example shows the case clearly.

Example 3	The 3rd and 7th term of a geometric progression are $\frac{1}{4}$ and 4 respectively. Find a) the common ratio b) the first term c) the general term		Explanation two general terms and only one a.
solution	a)	$T(n) = aR^{n-1}$ (1) $aR^2 = \frac{1}{\underline{\qquad}}$ (2) $aR^6 = \underline{\qquad}$ (3) (2) ÷ (1): $R^4 = \underline{\qquad}$ $R = \pm(\underline{\qquad})^{\frac{1}{4}}$ $= \underline{\qquad}$ or $\underline{\quad}\,\underline{\quad}$ #	
	b)	Subs. R into (2): For $R = 2$ $a(\underline{\qquad})^2 = \frac{1}{\underline{\qquad}}$ $a = \underline{\qquad}^{-\underline{\quad}}$ # For $R = -2$ $a(-\underline{\qquad})^2 = \frac{1}{\underline{\qquad}}$ $a = \underline{\qquad}^{-4}$ #	
	c)	By (1): For $R = 2$ $T(n) = 2^{-4}(\underline{\qquad})^{n-1} = \underline{\qquad}^{n-5}$ # For $R = -2$ $T(n) = 2^{-4}(-\underline{\qquad})^{n-1} = (-\underline{\qquad})^{n-5}$ #	

2 Geometric means

Definition:

The intermediate terms between two terms of a geometric progression are called geometric **means** between the two terms.

Example 4

Progression	Between	geometric means
2, 4, 8, 16, 32, …	2, 32	4, 8, 16
1, -3, 9, -27, 81, …	1, -27	-3, 9
4, 16, 64, 256, 1024, …	4, 1024	16, 64, 256

Insert n geometric means between a and b. a becomes the $\underline{1^{st}}$ term and b the $\underline{(n+2)^{th}}$ term.

Example 5	Insert 3 geometric means between 10 and 160.	Explanation
Solution	$T(n) = aR^{n-1}$ $\_\_ = \_\_R^4$ $R = \pm\left(\dfrac{\_\_\_}{\_\_\_}\right)^{\frac{1}{4}}$ $= \_\_$ or $- \_\_$ For R = 2, the geometric means are __, __ and __ # For R = -2, the geometric means are - __, __ and – __ #	160 is the 5th term.

The geometric mean of two numbers can be found by the same method. Again, a simpler formula exists. If *a*, *b*, *c* are three consecutive terms in a geometric progression, then *b* is the geometric mean of *a* and *c*. By the definition of (F2)

$$\text{geometric mean} = \pm\sqrt{ac} \qquad \text{(F4)}$$

For example, the geometric mean of 2 and 72 are 12 or -12

3 Geometric Series

For a geometric progression, the series $T(1) + T(2) + T(3) + \cdots + T(n)$ is given by

$$S(n) = \frac{a(1-R^n)}{1-R} = \frac{a(R^n-1)}{R-1} \qquad \text{(F5)}$$

Example 6	Find the sum of the geometric series Find the sum of the geometric series a) 2 + 4 + 8 + …to 10 terms b) 3 + 12 + 48 + 192 + …+ 12288	Explanation
Solution	a) $S(n) = \dfrac{a(R^n-1)}{R-1}$ $S(10) = \dfrac{2(\_\_^{10}-1)}{\_\_-1}$ $= \_\_\_$ #	

	B	$T(n)=aR^{n-1}$ $\underline{\qquad}=3(\underline{\quad})^{n-1}$ $n-1=\dfrac{\log\left(\dfrac{\underline{\qquad}}{\underline{\quad}}\right)}{\log\underline{\quad}}$ $=\underline{\quad}$ $n=\underline{\qquad}$ $S(7)=\dfrac{3(\underline{\quad}^{7}-1)}{\underline{\quad}-1}$ $=\underline{\qquad}$ #	

GP is very often used in daily life. "Annuity Due" is an example.

Example 7	\$10000 is deposited in a bank on the first day of each month for 4 years and the interest rate is 8% per annum. Find the total amount accumulated at the end of the fourth year. (correct to the nearest dollar)	Explanation
Solution	$A=1000(1+\dfrac{\underline{\qquad}}{12})\underline{\quad}+\underline{\qquad}(1+\dfrac{\underline{\qquad}}{12})^{47}$ $+1000(1+\dfrac{\underline{\qquad}}{12})^{46}+\cdots+\underline{\qquad\quad}+\dfrac{\underline{\qquad}}{12})$ $=\dfrac{1000(1+\dfrac{0.08}{12})((1+\dfrac{\underline{\qquad}}{12})^{48}-1)}{(1+\dfrac{\underline{\qquad}}{12})-1}$ = \$ $\underline{\quad}\ \underline{\quad}$ #	Amount = contribution from 1st month + contribution from 2nd month + … + contribution from 48th month interest rate = 8% =0.08 Number of terms =48

Mortgage is another example that is related to GP

Example 8	Johm borrows one million from a bank. $\$x$ is paid to the bank on the last day of each month. His debt will be over after 4 years if the interest rate is 6% per annum. Find the amount value of x.	Explanation
Solution	$1000000(1+0.005)^{48}=x(1+0.005)^{47}+x(1+0.005)^{46}$ $+x(1+0.005)^{45}+\ldots+x$ $=\dfrac{x[(1+0.005)^{48}-1]}{(1+0.005)-1}$ $x=\underline{\quad}\ \underline{\quad}$ #	Debt = money saved. Number of terms =48

2. Sum to infinity of a Geometric Series

Here is another distinction between arithmetic progressions and geometric progressions:

usually we cannot sum infinite terms for any arithmetic progression (why?) However, it may be done for some geometric progressions. By (F5)

$$S(n) = \frac{a(1-R^n)}{1-R}$$

If $-1 < R < 1$, when n is large, R^n is close to zero. As $n \to \infty$, $R^n \to 0$.

The **sum to infinity** of the geometric series S(∞) is given by

$$S(\infty) = \frac{a}{1-R} \qquad (-1 < R < 1) \qquad \text{(F6)}$$

Example 9	Find the sum to infinity of the geometric series $12 + 4 + \frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \ldots$		Explanation
Solution	a)	$S(\infty) = \frac{a}{1-R}$ (1) $R = \frac{\_\_\_}{\_\_\_} = \frac{1}{\_\_\_}$ subs. R into (1): $S(\infty) = \frac{\_\_\_}{1-\frac{1}{\_\_\_}}$ $= \_\_\_\_\_\_$ #	