Unit 18: Addition and Multiplication Laws

Learning Objectives

The students should be able to

- identify mutually exclusive events
- identify independent events
- state the addition law and multiplication law of probability
- solve simple problems involving addition and multiplication of probabilities
- identify the use of probabilities in daily life

Addition and Multiplication Laws

1. Introduction

In the last unit, we solve the problems by counting of the number of outcomes. Here, we shall use formulae to calculate the probability of events that involves "and" and "or".

2. Addition Law

Mutually Exclusive events

If A and B cannot occur simultaneously, they are mutually exclusive events.

Formula:

If $E_1, E_2, E_3, \ldots, E_n$ are mutually excusive,

then $P(E_1 \text{ or } E_2 \text{ or } E_3 \text{ or } \ldots \text{ or } E_n) = P(E_1) + P(E_2) + P(E_3) + \ldots + P(E_n)$ (F1)

Example 1	In drawing a card from a pack of playing cards, A is the event that the card is a 'king', B is the event that the card is a 'queen', and C is the event that the card belongs to the 'heart' suit. (a) Discuss if the events A, B and C are mutually exclusive . (b) Find the probability of getting a 'king' . (c) Find the probability of getting a 'queen' . (d) Find the probability of getting a 'heart' . (e) Find the probability of getting a 'king' or a 'queen'. (f) Find the probability of getting a 'king' or a 'heart'.
Solution	a) A and B are mutually exclusive; (A and C) and (B and C) are not. b) P(king) = $\frac{}{52} = \frac{}{}$ # c) P(queen) = $\frac{}{52} = \frac{}{}$ # d) P(heart) = $\frac{}{52} = \frac{1}{}$ # e) P(king or queen) = P(king) + P(queen) = $\frac{}{13} + \frac{}{13} = \frac{}{13}$ # f) favourable outcomes = {♠K, ♣K, ♦K, 13 ♥s } P(king or heart) = $\frac{}{52} = \frac{}{13}$ #

Note that (f) cannot be calculated by (F1).

Example 2	In a soccer match, the probability that team A will win is $\frac{1}{4}$ and the probability that team B will win is $\frac{1}{3}$. Find the probability that a) team A or B will win the match. b) The two teams tie.
Solution	a) P(A win or B win) = P (A win) + P(B win) $= \frac{}{4} + \frac{}{3}$ $= \frac{}{12}$ # b) P(tie) = 1 - P(not tie) = 1 – P(A win or B win) $= 1 - \frac{}{12}$ $= \frac{}{12}$ #

3. Multiplication Law

Independent events

If whether A has occurred or not will not affect the probability of B's occurrence, they are independent events.

Formulae:

If $E_1, E_2, E_3, \ldots, E_n$ are independent,

then $P(E_1 \,\&\, E_2 \,\&\, E_3 \,\&\, \ldots \,\&\, E_n) = P(E_1)P(E_2)\,P(E_3) \ldots P(E_n)$ (F2)

For only two events:

If E_1 & E_2, are **NOT** independent,

then $P(E_1 \,\&\, E_2) = P(E_1)P(E_2$ after E_1 has occurred$)$ (F3)

3	A bag contains 3 white balls and 4 green balls. Another bag contains 2 yellow marbles and 3 blue marbles. One ball is chosen from the first bag and a marble is selected from the second bag randomly, find the probability that a white ball and a yellow marble are chosen.

Solution

P(a white ball and a yellow marble are chosen)

= P(1^{st} W & 2^{nd} Y)

= P(1^{st} W) P(2^{nd} Y)

$= \frac{}{3+} \times \frac{}{2+}$

$= \frac{}{35}$ #

Example 4

A bag contains 4 red and 6 green balls. Two balls are drawn successively at random. Given that A is the event that the first ball drawn is red, B is the event that the second ball drawn is red with replacement, and C is the event that the second ball drawn is red without replacement.

(a) Discuss if the events A, B and C are independent .

Find the probability that the two balls are red if

(b) they are drawn with replacement .

(c) they are drawn without replacement.

Solution

a) A and B are independent; (A and C) and (B and C) are not.

b) P(1^{st} R & 2^{nd} R) = $\frac{4}{10} \times \frac{}{10}$ = 0. #

c) P(1^{st} R & 2^{nd} R) = $\frac{4}{10} \times \frac{3}{9} = \frac{}{}$

Example 5

Bag A contains 3 red marbles and 2 green marbles. Bag B contains 5 green marbles only. One marble is drawn randomly from A and put into bag B. Then one marble is drawn from bag B randomly. Find the probability that the marble drawn from bag B is

a) red

b) green

Solution

a) P(red from bag B) = P(1^{st} red & 2^{nd} red)

= P(1^{st} red) P(2^{nd} red)

$= \frac{}{5} \times \frac{1}{6}$

= 0. #

b)

P(green from bag B) = P("1^{st} red & 2^{nd} green" or "1^{st} green & 2^{nd} green")

= P(1^{st} red) P(2^{nd} green) + P(1^{st} green) P(2^{nd} green)

$= \frac{3}{5} \times \frac{}{6} + \frac{2}{5} \times$

= 0. #

* simpler solution exist. Can you figure it out?

Example 6	A, B and C each shoots at a target. The probability that the target is hit by A, B and C are $\frac{1}{2}, \frac{2}{3}$ and $\frac{3}{4}$ respectively. Find the probability that (a) all of them hit the target. (b) the target is hit. (c) only of them hits the target.
Solution	a) P(all hit the target) = P(A hits & B hits & C hits) $= \frac{1}{2}\times\frac{2}{3}\times\frac{}{4} = 0.$ # b) P(the target is hit) = 1 – P(the target is not hit) = 1 – P(A fails & B fails & C fails) $= 1-(1-\frac{}{2})\times(1-\frac{}{3})\times(1-\frac{}{4})$ $= \frac{}{12}$ # c) P(only one hits) = P(“A hit & B fails & C fails) or “A fails & B hit & C fails” or “A fails & B fails & C hit”) $= \frac{1}{2}\times\frac{1}{3}\times\frac{}{4}+\frac{1}{2}\times\frac{2}{3}\times\frac{1}{4}+\frac{}{2}\times\frac{1}{3}\times\frac{}{4}$ $= 0.$ #

Finally, we are going to introduce another useful formula

P(A or B) = P(A) + P(B) – P(A &B) (F4)

- F4 is true for only two events
- F4 is a general formula: A and B has no constraint. (cf for F1 to be true, the events must be exclusive)

Example 7	In a class of 40 students, 22 of them have exempted Mathematics; 7 of them have exempted Computing Studies whereas 15 of them have to study both subjects. If a student is picked at random from the class, find the probability that (d) he/she has exempted Mathematics; (e) he/she has exempted at least one subjects; (f) he/she has exempted both subjects; (g) he/she has exempted Computing Studies but not Mathematics.

Solution	d) P(exempted Mathematics) = $\frac{}{40}$ = 0. # e) P(exempted at least one subjects) = 1 – P(study both subjects) = 1 - $\frac{}{40}$ = 0. # f) P(exempted Mathematics or exempted computing studies) = P(exempted Mathematics) + P(exempted computing) - P(exempted Mathematics & exempted computing studies) 0.625 = 0.55 + $\frac{}{40}$ - P(exempted both subjects) P(exempted both subjects) = 0.075# g) P(exempted Computing Studies only) = P(exempted computing) – P(exempted both subjects) = $\frac{}{40} - 0.075$ =0. #

- Find the no. of students that have exempted both subjects first is another approach.

Example 8	Janet and Peter take turns to toss a balanced coin until a "Head" turns up. The first one to toss a "Head" wins. If Janet starts first, find the probability that Janet will win.
Solution	P(Janet wins) = P((Janet wins at 1st time) or (Janet wins at 3rd time) or (Janet wins at 5th time) or …) = P(H or TTH or TTTTH …T^{2n}H …) $= \frac{1}{2} + (\frac{}{2})^3 + (\frac{}{2})^5 + \ldots$ $= \dfrac{\frac{1}{2}}{1 - \frac{}{4}}$ $= \frac{}{3}$ #