Unit 22 : Equation of a circle and its intersection with a straight line

Learning Objectives

Students should be able to:

- Recognize the standard form of a circle
- Determine the standard form of a circle when its center and radius are given
- Recognize the general form of a circle
- Determine the center and radius of a circle when its general form is given
- Determine the equation of a circle passing through three given points.
- Determine the intersection point(s) between a circle and a line
- Determine the number of intersection point(s) between a circle and a line
- Determine the equation of the tangent to a circle

1. The Standard Form of a Circle

A circle is the locus of the point which moves so that its distance from a fixed point is constant.

The fixed point is called the centre and the constant distance the radius of the circle (as shown in Fig. 1a).

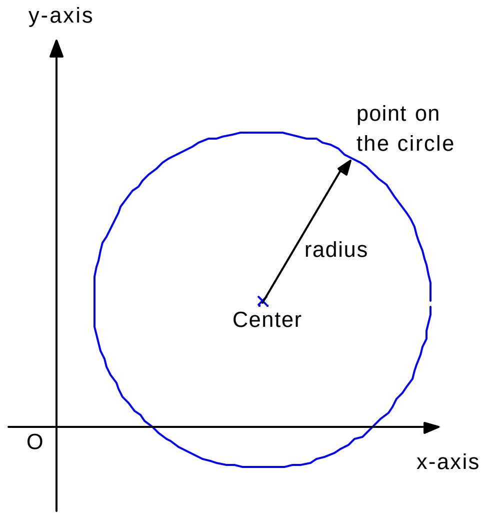


Fig. 1a

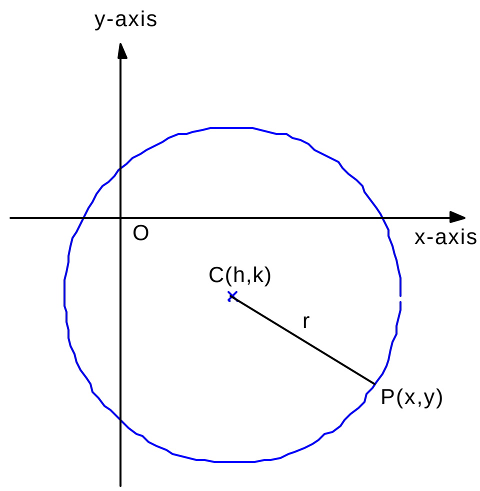


Fig. 1b

If a circle with center at C(h, k) and radius r, the distance of P(x, y), a point on the circle, from centre C(h, k) is equal to the radius r.

From Fig. 1b,

$$PC = \sqrt{(x-h)^2 + (y-k)^2} = r$$

or $$\boxed{(x-h)^2 + (y-k)^2 = r^2}$$

This equation is called the standard form of the circle with centre at C(h, k) and radius r.

Example 1:

Find the equation of the circle with centre at (3,2) and radius 5.

Solution:

The equation of the circle is

$$(x-3)^2 + (y-2)^2 = 5^2$$

or $x^2 + y^2 - 6x - 4y - 12 = 0$

Example 2:

Find the equation of the circle with centre at (2, -1) and radius $\sqrt{7}$.

Solution:

The equation of the circle is

$$(x-2)^2 + (y+1)^2 = \sqrt{7}^2$$

or $x^2 + y^2 - 4x + 2y - 2 = 0$.

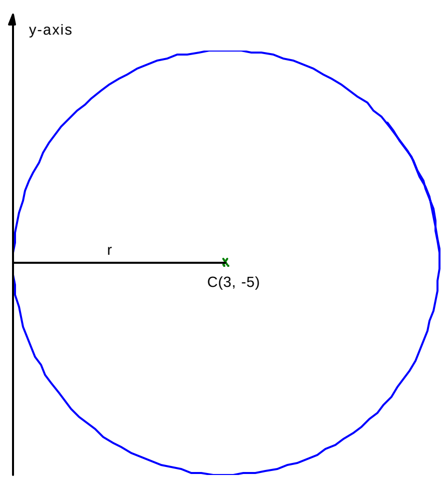
Example 3: Find the equation of the circle with centre at C(2, 5) and passes through A(-2, 2).

Solution: Radius CA$=\sqrt{4^2 + 3^2} = 5$

Hence equation of the circle is:

$$(x-2)^2 + (y-5)^2 = 5^2$$

or $x^2 + y^2 - 4x - 10y + 4 = 0$.



Example 4: Find the equation of the circle centered at (3, -5) and tangential to the y-axis.

Solution: Radius $r = 3$

Hence equation of the circle is:

$(x-3)^2 + (y+5)^2 = 3^2$

or $x^2 + y^2 - 6x + 10y + 25 = 0$.

2. The General Form of a Circle

The stand form of a circle is $(x-h)^2 + (y-k)^2 = r^2$, expanding this equation gives $x^2 + y^2 - 2hx - 2ky + (h^2 + k^2 - r^2) = 0$ … … (*)

From this and the results we have worked out in the previous examples, we realize that the equation of a circle is quadratic in both x and y but without xy term. Besides, the coefficients of x^2 and y^2 terms are equal (usually set to unity). We therefore conclude that every circle can be written in the form

$$x^2 + y^2 + dx + ey + f = 0 \quad \dots \dots \quad (**)$$

and (**) is called the general form of a circle.

If (*) and (**) represent the same circle, then

$$\begin{cases} d = -2h \\ e = -2k \\ f = h^2 + k^2 - r^2 \end{cases} \quad \text{or} \quad \begin{cases} h = \dfrac{-d}{2} \\ k = \dfrac{-e}{2} \\ r = \sqrt{(\dfrac{d}{2})^2 + (\dfrac{e}{2})^2 - f} \end{cases}$$

Hence $x^2 + y^2 + dx + ey + f = 0$ represents a circle with centre at $(\frac{-d}{2}, \frac{-e}{2})$ and radius $r = \sqrt{(\frac{d}{2})^2 + (\frac{e}{2})^2 - f}$.

Example 5: Find the centre and radius of the circle $x^2 + y^2 - 6y = 0$;

Solution:

$h = \frac{-0}{2} = 0$, $k = \frac{-(-6)}{2} = 3$

and $r = \sqrt{0^2 + (-3)^2 - 0} = 3$

hence centre C = (0, 3) and radius r = 3.

Example 6: Find the equation of the circle which passes through the points (4, 1), (5, 0) and (-2, -7).

Solution: Let the equation be $x^2 + y^2 + dx + ey + f = 0$

Then 16+1+4d+e+f = 0

4d+e+f = -17 … …(1)

25+0+5d+0e+f = 0

5d+0e+f = -25 … …(2)

4+49-2d-7e+ f = 0

-2d-7e+ f = -53 … …(3)

(2)-(1) : d-e = -8 ……………………..(4)

(2)-(3) : 7d+7e = 28

d+e = 4 ……………………..(5)

(4)+(5) : d = -2

Sub. d = -2 into (5) : e = 6

Sub. d = -2 into (2) : f = -15

Hence the equation of the circle is $x^2 + y^2 - 2x + 6y - 15 = 0$.

3. Intersection between a circle and a line

Consider a line L: $y = ax + b$

and a circle C: $x^2 + y^2 + dx + ey + f = 0$,

by substituting L into C,

$$x^2 + (ax+b)^2 + dx + e(ax+b) + f = 0 \quad \dots \dots (*)$$

equation (*) is quadratic and gives the x-coordinates of the points in which the line L meets the circle C.

Example 7 : Find the point(s) of intersection of the line 4 x – 3 y + 16 = 0 and the circle $x^2 + y^2 - 6x - 2y - 15 = 0$

Solution: $4x - 3y + 16 = 0$

$$x = \frac{3}{4}y - 4 \qquad (1)$$

$$x^2 + y^2 - 6x - 2y - 15 = 0 \qquad (2)$$

Subs (1) into (2) :

$$\left(\frac{3}{4}y - 4\right)^2 + y^2 - 6\left(\frac{3}{4}y - 4\right) - 2y - 15 = 0$$

$$\frac{25}{16}y^2 - \frac{25}{2}y - 25 = 0$$

$$y^2 - 8y + 16 = 0$$

$$(y - 4)^2 = 0$$

$y = 4$ (repeated)

When $y = 4$, $x = \frac{3}{4}(4) - 4 = -1$

The point of intersection is (-1, 4)

4. <u>Condition for a line to be a tangent to a circle</u>

Let Δ denote the discriminant of equation (*),

In case I: there are 2 distinct intersection points, $\Delta > 0$;

In case II: there is 1 intersection point, $\Delta = 0$;

In case III: there is no intersection point, $\Delta < 0$.

Note that if a line L intersects (touches) a circle C at 1 point, this line is a **tangent** to the circle. Hence the condition that L is a tangent to C is $\Delta = 0$.

<u>Example 8</u>: (a) Find the point(s) of intersection between the circle C : $x^2 + y^2 - 6x - 18y + 65 = 0$ and y-axis.

(b) Show that there is no intersection point between C and x-axis.

Solution: (a) $x^2 + y^2 - 6x - 18y + 65 = 0$ (1)

$x = 0$ (2)

Subs (2) into (1), $y^2 - 18y + 65 = 0$

$(y - 5)(y - 13) = 0$

$y = 5$ or $y = 13$

C cuts the y – axis at (0, 5) and (0, 13).

(b) Substitute $y = 0$ into (1) ,

$$x^2 - 6x + 65 = 0 \qquad (3)$$

The discriminant of (3) : $\Delta = (-6)^2 - 4(1)(65) = -224 < 0$

There is no intersection between circle C and the x-axis.

<u>Example 9</u>: Find the value(s) of m if the line $y = mx$ is a tangent to the circle $x^2 + y^2 - x - 2y + 1 = 0$.

Solution: Substitute $y = mx$ into $x^2 + y^2 - x - 2y + 1 = 0$, we get

$$x^2 + m^2x^2 - x - 2mx + 1 = 0$$

$$(m^2 + 1)x^2 - (2m + 1)x + 1 = 0$$

$\Delta = 0$: $(2m + 1)^2 - 4(m^2 + 1)(1) = 0$

$$4m = 3 \qquad \text{or} \qquad m = \frac{3}{4}$$

The equation of the tangent is $y = \frac{3}{4}x$.