# Unit 3: Factorization of simple expressions

## Learning Objectives

The students should be able to:

- Factorize simple quadratic expressions by
  - (a) extraction of a common factor
  - (b) formulae
  - (c) cross-product method
- use factor theorem to factorize expressions

# Factorization of Quadratic expressions

**Introduction:**

The aim of factorization is to express a sum of terms as a product of terms. The technique of factorization is used in solving equations (see the unit of Quadratic Equations). We shall focus on the technique of factorization of quadratic expressions.

The basic methods:

1. Extraction of common terms
2. Factorization by formulae
3. Factorization by the cross product method

**Factorization by the extraction of a common factor:**

- The key point is to identify the common factor.

If $f$ is a common factor of the terms, $t_1$, $t_2$ … $t_n$ then ,

$$t_1 + t_2 + t_3 = f\left(\frac{t_1}{f} + \frac{t_2}{f} + \frac{t_3}{f}\right), \ldots, t_1 + t_2 + \ldots + t_n = f\left(\frac{t_1}{f} + \frac{t_2}{f} + \cdots + \frac{t_n}{f}\right)$$

| E.g. 1 | Factorize the following expression if possible.<br>a) $4xy - 8x^2y^3$<br>b) $a(b+3) + 3(3+b)$<br>c) $a(b-3) + 3(3-b)$ | | Explanation. |
|---|---|---|---|
| Soln | a) | $4xy - 8x^2y^3$<br>$= 4xy(1-2xy^2)_{\#}$ | $f = 4xy$.<br>$\frac{4xy}{4xy} = 1, \frac{-8x^2y^3}{4xy} = -2xy^2$ |
| | b) | $a(b+3) + 3(3+b)$<br>$= (\_\_\_\_\_\_)(a+3)_{\#}$ | $f = b+3$. |
| | c) | $a(b-3) + 3(3-b)$<br>$= a(b-3) - 3(\_\_\_\_\_\_)$<br>$= (a-3)(\_\_\_\_\_\_)_{\#}$ | (3-*b*)=-1(*b*-3)<br>Note that expressions involving (*x*-*y*) and (*y*-*x*) frequently occur. |

**Factorization by formulae:**

When the quadratic term and the term that does not contain $x$ are perfect square, the following formulae may be useful.

$(x+a)^2 = x^2 + 2ax + a^2$ (F1)

$(x-a)^2 = x^2 - 2ax + a^2$ (F2)

$(x+a)(x-y) = x^2 - a^2$ (F3)

| | | |
|---|---|---|
| E.g. 2 | Factorize $x^2+2x+1$ | Explanation |
| Solution | $x^2+2x+1$<br>$= x^2+2x+\_\_\_^2$<br>$= (x+\_\_\_)^2$ | $1 = 1^2$<br>(F1) |
| E.g. 3 | Factorize $4x^2-12x+9$ | Explanation |
| Solution | $4x^2-12x+9$<br>$= (2x)^2 - 2(2x)(3) + 3^2$<br>$= (2x-3)^2$ | $4x^2 = (2x)^2$,<br>$9 = 3^2$<br>(F2) |
| E.g. 4 | Factorize $x^2-y^2$ | Explanation. |
| Solution | $x^2-y^2$<br>$= (x+y)(\_\_\_-\_\_\_)$ | (F3) |
| E.g. 5 | Factorize $x^4-16y^4$ | Explanation. |
| | $x^4-16y^4$<br>$= (\_\_\_^2)^2 - (\_\_\_^2)^2$<br>$= (x^2+4y^2)(x^2-(\_\_\_)^2)$<br>$= (x^2+4y^2)(x^2+\_\_\_)(x-\_\_\_)$ | Always check if the formulae can be used. |
| E.g. 6 | Factorize $8x^2-24x+18$ | Explanation |
| Solution | $8x^2-24x+18$<br>$= 2(4x^2-\_\_\_x+\_\_\_)$<br>$= 2[(2x)^2-2(2x)(3)+\_\_\_^2]$<br>$= 2(\_\_\_x-\_\_\_)^2$ | Always extract the common factors first. |

**Factorization by the cross-product method:**

It is easy to factorize $x^2 + bx + c$.

Steps:

1. List the factor pairs ($d$, $e$) of $c$ by mental calculation. (You may write them down on a piece of paper)
2. Choose the pair if $d + e = b$. If $d + e = -b$, choose ($-d$, $-e$)
3. $x^2 + bx + c = (x + d)(x + e)$

| | | |
|---|---|---|
| E.g. 7 | Factorize $x^2 + 5x + 6$ | Explanation. |
| Solution | $x^2 + 5x + 6$<br>$= (x + \_\_\_)(x + \_\_\_)$ | $b = 5$, $c = 6$.<br>List of factors of 6:<br>(1, 6): sum $\neq b$.<br>(2, 3): sum $= b$, yes. |
| E.g. 8 | Factorize $2x^2 + 12x + 16$ | Explanation. |
| Solution | $2x^2 + 12x + 16$<br>$= 2(x^2 + 6x + \_\_)$<br>$= 2(x + \_\_\_)(x + \_\_\_)$ | Always extract the common factors first.<br>$b = 6$, $c = 8$.<br>List of factors of 8:<br>(1, 8): sum $\neq b$.<br>(2, 4): sum $= b$, yes. |
| E.g. 9 | Factorize $x^2 + x - 6$ | Explanation. |
| Solution | $x^2 + x - 6$<br>$= (x - \_\_\_)(x + \_\_\_)$ | $b = 1$, $c = -6$<br>List of factors of –6:<br>(1, -6): sum $\neq b$.<br>(2, -3): sum $= -b$, choose (-2,3) |

**Factorization of $ax^2 + bx + c$ by the cross product method**

Steps:

1. List the factor pairs ($m$, $n$) of $a$ with $m > 0$.
2. List the factor pairs ($d$, $e$) of $c$ with $d > 0$.
   (You may write them down on a piece of paper)
3. Choose the pairs if $me + nd = b$. If $me + nd = -b$, choose ($-d$, $-e$).
4. $ax^2 + bx + c = (mx + d)(nx + e)$

Remark:

This method is straight forward, but trials and errors are required. It may take a long time to get the correct factor pairs. Students not good at mental calculation would find the method not easy.

E.g. 10 Factorize $2x^2 - x - 6$.

$2x^2 - x - 6 = (x-2)(\underline{\quad}x+\underline{\quad})$

| | | |
|---|---|---|
| 1 | -2 | |
| 2 | 3 | |
| $1(3)+2(-2) = -1$ | | |

Explanation:
$a = 2 = 1(2)$
$b = -1$

$c = -6 = -2(3)$
$me + nd = b$

E.g. 11 Factorize $6x^2 - 9x - 6$.

$6x^2 - 9x - 6$

$= 3(2x^2 - \underline{\quad}x - \underline{\quad})$

$= 3(2x + \underline{\quad})(x - \underline{\quad})$

| | |
|---|---|
| 2 | 1 |
| 1 | -2 |
| $2(-2)+1(1) = -3$ | |

Explanation:
Always extract common factor

**Factor Theorem**

Let $f(x)$ represents a function (an expression) of $x$. Then $f(a)$ is the value of the function when $x$ is substituted by $a$.

E.g. 12 $f(x) = 2x^2 - x - 6$. Find $f(1)$ and $f(-2)$.

$f(1) = 2(1)^2 - \underline{\quad\quad} - 6 = -\underline{\quad\quad}$

$f(-2) = 2(-\underline{\quad})^2 - (-\underline{\quad}) - 6 = \underline{\quad\quad}$

Explanation:
$a = 6 = 2(4)$

* Factor theorem states that if $f(a) = 0$, then $x - a$ is a factor of $f(x)$. We may use this theorem for factorization.

E.g. 13 Factorize $x^3 + 6x^2 + 11x + 6$.

Let $f(x) = x^3 + 6x^2 + 11x + 6$

$f(-1) = (\underline{\quad})^3 + 6(\underline{\quad})^2 + 11(\underline{\quad}) + 6 = 0$

$(x + 1)$ is a factor of $x^3 + 6x^2 + 11x + 6$

After long division, we have

$x^3 + 6x^2 + 11x + 6$

$= (x + 1)(x^2 + \underline{\quad}x + 6)$

$= (x + 1)(x + \underline{\quad})(x + \underline{\quad})$

Explanation:
$a = 6 = 2(4)$