Unit 4: Quadratic equations in one unknown

Learning Objectives

The students should be able to:

- Define a quadratic equation and its solutions (roots)
- Solve quadratic equations by factorization
- Solve quadratic equations by the quadratic formula
- Determine the nature of the roots by the discriminant or the quadratic graph
- Find the sum of roots and the product of roots
- Solve practical problems leading to quadratic equations
- Form quadratic equations with given roots

Quadratic equations in one unknown

1. What is a quadratic equation?

A quadratic equation is an equation that can be written as $ax^2 + bx + c = 0$, where a, b and c are constants and $a \neq 0$.

Eg. 1	Determine if the following equations are quadratic equations	Explanation
	a) $2x^2 + 3x - 4 = 0$	Yes, form correct, $a \neq 0$
	b) $2x^2 + 3x = 0$	___, form correct , $a \neq 0$, c =0
	c) $2x^2 - 4 = 0$	___, form correct , $a \neq 0$, b =0
	d) $2x^2 + 3x - 4 = x^2 - x + 3$	___, after rearrangement: form correct , $a \neq 0$
	e) $2x^2 + 3x - 4 = 2(x-1)^2$	___, after rearrangement: a = 0
	f) $2x^2 + 3\sqrt{x} - 4 = 0$	___, should not contain non-integral power of x.

A solution (root) of an equation is a real number that satisfies the equation when it replaces the variable of the equation. In other words, LHS=RHS after substitution.
e.g. 3 is a solution of $x^2 - 9 = 0$ because $3^2 - 9 = 0$ = the RHS.
e.g. 2 is not a solution of $x^2 - 9 = 0$ because $2^2 - 9 = -5 \neq$ the RHS.

2. How to solve a quadratic equation?

2.1 By factorization

- If $xy = 0$ then either $x = 0$ or $y = 0$.

Similarly, if $ax^2 + bx + c = 0$ can be written as $(dx + e)(fx + g) = 0$, then either $dx + e = 0$ or $fx + g = 0$.
Both of the equations may be solved readily.

Eg. 2	Solve $x^2 + 3x + 2 = 0$	Explanation
Solution	$x^2 + 3x + 2 = 0$	
	$(x+1)(x+2) = 0$	Factorization
	$x + 1 = 0$ or $x + 2 = 0$	Optional.
	$x = -1$ or $x = -2_{\#}$	

Eg. 3	Solve $x^2 - 8 = 1$	
Solution	$x^2 - 8 = 1$ $x^2 - 9 = 0$ $(x + \_\_\_)(x - \_\_\_) = 0$ $x = -\_\_\_ \ or\ \_\_\__{\#}$	Convert the equation into the standard form Factorization.
Eg. 4	Solve $6x^2 + 23x + 21 = 0$	
Solution	$6x^2 + 23x + 21 = 0$ $(2x + \_\_\_)(3x + \_\_\_) = 0$ $x = \frac{-\_\_\_}{2} \ or\ \frac{-7}{\_\_\_}_{\#}$	Factorization.

There are a few drawbacks of the factorization method. Firstly, when $a \neq 1$, the factorization may not be done easily. Secondly, there are quadratic equations that cannot be factorized without using surds; $x^2 + x - 1 = 0$ is an example.

2.2 By formula

- $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Eg. 5	Solve $x^2 + 3x + 2 = 0$	Explanation
Solution	$x^2 + 3x + 2 = 0$ $x = \frac{-3 \pm \sqrt{\_\_\_^2 - 4(\_\_)(2)}}{2(\_\_\_)}$ $= \frac{-\_\_\_ \pm 1}{\_\_\_}$ $= -\_\_\_\_ \ or\ -\_\_\_\__{\#}$	$a = 1$, $b = 3$, $c = 2$.
Eg. 6	Solve $x^2 - 8 = 1$	
Solution	$x^2 - 8 = 1$ $x^2 - 9 = 0$ $x = \frac{0 \pm \sqrt{-4(1)(-\_\_\_)}}{2(\_\_\_)}$ $= \_\_\_ \ or\ -\_\_\__{\#}$	$a = 1$, $b = 0$, $c = -9$.
Eg. 7	Solve $6x^2 + 23x + 21 = 0$	
Solution	$6x^2 + 23x + 21 = 0$ $x = \frac{-\_\_\_ \pm \sqrt{\_\_\_^2 - 4(\_\_\_)(\_\_\_)}}{2(\_\_\_)}$ $x = \frac{-3}{\_\_\_} \ or\ \frac{-\_\_\_}{3}_{\#}$	

Eg. 8	Solve $x^2+x-1=0$	
Solution	$x^2+x-1=0$ $x=\dfrac{-1\pm\sqrt{1^2-4(\_\_\_)(-1)}}{2(\_\_\_)}$ $=\dfrac{-1-\sqrt{\_\_\_}}{2}$ *or* $\dfrac{-1+\sqrt{\_\_\_}}{\_\_\_}$ #	
Eg. 9	Solve $x^2+x+1=0$	
Solution	$x^2+x+1=0$ $x=\dfrac{-\_\_\_\pm\sqrt{1^2-4(\_\_\_)(\_\_\_)}}{2(\_\_\_)}$ $=\dfrac{-1\pm\sqrt{-\_\_\_}}{\_\_\_}$ $\because \sqrt{-3}$ is not real, there are no real roots.	

3. Solve practical problems leading to quadratic equations

Step 1: Let the unknown variable be x, say.

Step 2: Set up a quadratic equation in x according to the given conditions.

Step 3: Solve the quadratic equation to find the solutions.

Step 4: Check if the value of the solution is valid and reject invalid values.

Eg. 10	The difference between two numbers is 6 and their product is 247. Find the two numbers.	
Solution	Let the smaller number be x. Then the larger number would be x + 6. $x(x+\_\_\_)=247$ $x^2+\_\_\_x-247=0$ $(x-\_\_\_)(x+\_\_\_)=0$ $x=\_\_\_$ *or* $-\_\ \_\_$ The two numbers are (13, ___) or (-___, -19)	Step 1 Step 2 Step 3
Eg. 11	The length of a rectangle is 6cm longer than its width. The area of the rectangle is 16cm^2. Find the length of the rectangle.	
Solution	Let the length be x. Then the width would be x - 6. $x(x-\_\_\_)=16$ $x^2-\_\_\_x-16=0$ $(x-\_\_\_)(x+\_\_\_)=0$ x = ___cm (x < 0 is rejected)	

Eg. 12	The speed of the water current is $x$km/hr. The speed of a boat in still water is x^2km/hr. After 1.5 hours upstream and 1 hour downstream, the boat has moved 26km. a) Write an equation in x b) Find x (correct to 2 dp)	
Solution	a) Total distance = distance travelled upstream + distance travelled downstream (1) Distance = speed $\times$ time (2) By (1) and (2): $26=(x^2-\_\_\_)(1.5)+(x^2+\_\_\_)(1)$ $2.5x^2-\_\_\_\,x-26=0$ $5x^2-x-\_\_\_=0$ b) $x=\dfrac{1\pm\sqrt{(-\_\_\_)^2-4(5)(\_\_\_)}}{2(\_\_\_)}$ x = ______ km/hr ($x<0$ is rejected)	Step 2, the key equation. Step 2 Step 3 Step 4.

4. **Nature of roots**

4.1 Discriminant and the number of roots of a quadratic equation

The discriminant, Δ(read as delta), is defined as

$$\Delta=b^2-4ac$$

Its value tells the number of distinct real roots of a quadratic equation.

$$\because\ x=\frac{-b\pm\sqrt{\Delta}}{2a}=-\frac{b}{a}\pm\frac{\sqrt{\Delta}}{2a},$$

$\therefore$ $\Delta>0 \iff$ 2 distinct real roots

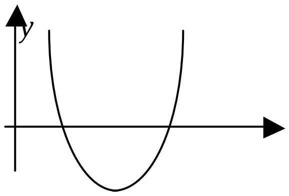
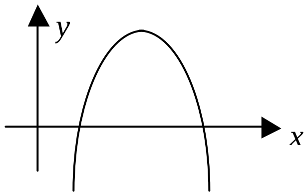
$\Delta=0 \iff$ 2 equal real roots or one distinct real root

$\Delta<0 \iff$ no real root

Eg. 13	Determine the nature of the roots of a) $x^2+2x-1=0$ b) $x^2+2x=0$ c) $x^2+2x+1=0$ d) $x^2+2x+2=0$	
Solution	a) $\Delta=2^2-4(\_\_\_)(-\_\_\_)=\_\_\_\_>\_\_\_$ 2 real roots b) $\Delta=2^2-4(\_\_\_)(\_\_\_)=\_\_\_>\_\_\_$ 2 real roots c) $\Delta=2^2-4(\_\_\_)(\_\_\_)=0$ equal root d) $\Delta=2^2-4(\_\_\_)(\_\_\_)=-\_\_\_<0$ no real root	

4.2 The shape of a quadratic graph

If we plot $y = ax^2 + bx + c$, then we will have two cases:

$a > 0$, the graph open upwards	$a < 0$, the graph open downwards
	

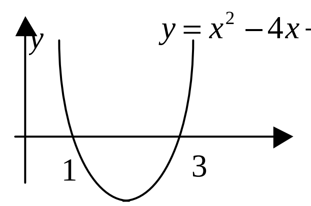
The roots of $ax^2 + bx + c = 0$ are the x-values of the points with $y = 0$.

$\Rightarrow$ the roots of $ax^2 + bx + c = 0$ are the x-intercepts of the graph $y = ax^2 + bx + c$

$\Rightarrow$ the roots $ax^2 + bx + c = 0$ can be read from the graph $y = ax^2 + bx + c$.

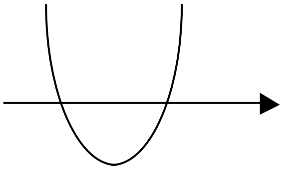
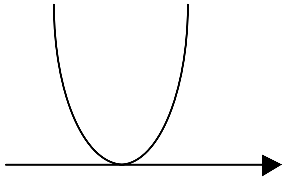
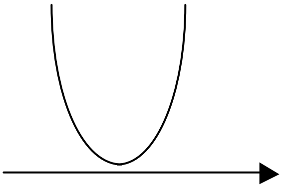
Example: Read from the graph to fill in the following table.

Condition	$y > 0$	$y = 0$
Value of x	$x >$ ___ or $x <$ ___	___, ___
Condition	$x^2 - 4x + 3 < 0$	$x^2 - 4x + 3 = 0$
Value of x	___ $< x <$ ___	___, ___



4.3 Discriminant and the number of x-intercepts of the quadratic graph

Because each x-intercept is a root, we have

	$\Delta > 0 \Leftrightarrow$ two x-intercepts of the quadratic graph	$\Delta = 0 \Leftrightarrow$ one x-intercept of the quadratic graph	$\Delta < 0 \Leftrightarrow$ no x-intercept of the quadratic graph
$a>0$			
$a<0$