Unit 9 : Proportion and Variation

Learning Objectives

Students should be able to

- apply the k-method to solve various problems concerning ratio and proportion
- state the meaning of direct variation
- solve problems involving direct variation
- state the meaning of inverse variation
- solve problems involving inverse variation
- describe joint variation as a product of several quantities
- solve problems involving joint variation
- express partial variation as a sum of several quantities
- solve problems involving partial variations

1. Proportion

If $a : b = c : d$, then a, b, c and d are said to be in proportion.

Example 1

x, 4, 5 and 8 are in proportion, find x.

Solution

$\frac{x}{4} = \frac{5}{8}$ is equivalent to $8x = 4 \times 5$,

so $x = 2.5$.

Example 2

Given $\frac{a}{b} = \frac{c}{d}$, show that

(a) $\frac{a+b}{b} = \frac{c+d}{d}$

(b) $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

Solution

Let $a = ck$
$b = dk$ for some constant k

(a) $\frac{a+b}{b} = \frac{ck+dk}{dk} = \frac{c+d}{d}$

$\therefore \frac{a+b}{b} = \frac{c+d}{d}$

(b) $\frac{a+b}{a-b} = \_\_\_\_\_\_ = \_\_\_\_\_\_$

$\therefore \frac{a+b}{a-b} = \frac{c+d}{c-d}$

Example 3

(a) If $\frac{p}{q} = \frac{r}{s}$, show that $\frac{p-q}{p+q} = \frac{r-s}{r+s}$

(b) Hence solve the equation $\frac{4y^2+y-2}{4y^2-y-2} = \frac{y+2}{y-2}$.

Solution : (a) Let $\frac{p}{q} = \frac{rk}{sk}$, $p=\quad k$, $r = \quad k$

L.H.S. = $\frac{p-q}{p+q} = \frac{\quad}{\quad}$

$=$

$=$ R.H.S.

.

(b) $\frac{(4y^2+y-2)-(4y^2-y-2)}{(4y^2+y-2)+(4y^2-y-2)} = \frac{(y+2)-(y-2)}{(y+2)+(y-2)}$

$\frac{2y}{y^2 -} = \frac{\quad}{2y}$

$4y^2 = \quad y^2 - 16$

$16 = \quad y^2$

$4 = \quad y^2$

$y = \pm$

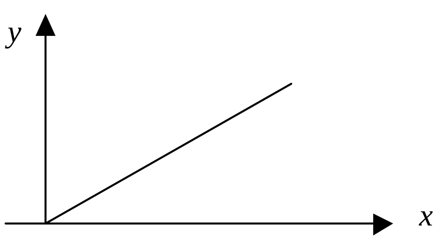
2. Review on Direct Variation

If two quantities x and y are so varying that the ratio of y to x is always a constant, then y is said to vary directly as x. The relation may be written $y \propto x$ and read as 'y is proportional to x' or 'y varies directly as x'.

Hence if $y \propto x$

then $\frac{y}{x} = k$ (k = constant)

or $y = kx$.



When y varies directly as x, the equation between y and x is $y = kx$ where k is the variation constant. The graph of **y against x is a straight line passing through the origin** and its slope represents the variation constant k.

Example 4

The area of an isosceles right-angled triangle varies directly as square of the length of the side. If the area is 18 cm^2 when the length of the side is 6 cm,

(a) express the area in terms of the length of the side;

(b) find the area of an isosceles right-angled triangle with side 5 cm.

Solution

(a) Let A be the area of the triangle and l be the length of the side.
Since A varies directly as the square of the length, $A \propto l^2$ and

$A = kl^2$

When $l = 6$ cm, $A = 18$ cm^2,

$= k(\quad)^2$

$k =$

Hence $A = \frac{1}{2}l^2$

(b) When $l = 5$,

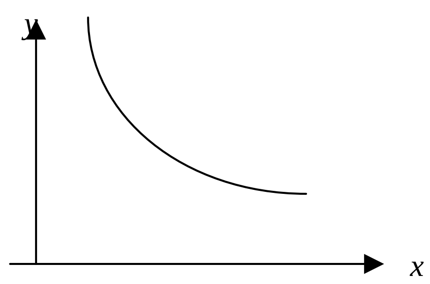
$A = \frac{1}{2}(\quad)^2 = \frac{\qquad}{2} = \qquad$ cm^2

3. Review on Inverse Variation

A quantity y is said to **vary inversely as** or is **inversely proportional to** another quantity x if y varies directly as $\frac{1}{x}$.

In symbol, if $y \propto \frac{1}{x}$ then $y = \frac{k}{x}$ where k is the variation constant,
or $xy = k$.

Hence when y varies inversely as x, the product xy is a constant



When y varies inversely as x, the equation between y and x is $y = k\frac{1}{x}$. The graph of y against $\frac{1}{x}$ is a straight line passing through the origin and its slope represents the variation constant k.

Example 5

If y varies inversely as x and $y = 4$ when $x = 30$.

(a) Find the relation between y and x.

(b) Find the value of y when $x = 3$.

Solution

(a) Since $y \propto \frac{1}{x}$, we can write $y = k\frac{1}{x}$, where k is the variation constant.

It is given that when $x = 30, y = 4$

$4 = k\frac{1}{\quad}$

$k =$

Hence the relation between x and y is $y = \frac{\quad}{x}$.

(b) When $x = 3$, $y = \quad (\quad) =$

4. Joint Variation

A quantity is said to **vary jointly** as several other quantities, if it varies as the *product* of these quantities.

z varies jointly as x and y, if $z = k\,xy$ where k is the variation constant.

A joint variation can also be a combination of direct and inverse variations. If a varies directly as b and inversely as c, then $a=k\frac{b}{c}$ where k is the variation constant.

Example 6

w varies jointly as u and $\sqrt{v}$. If $w = 9$ when $u = 3$ and $v = 36$, find

(a) the variation constant

(b) w when $u = 10$ and $v = 81$.

Solution

(a) $w \; a \; u\sqrt{v}$

$w = \quad u\sqrt{v}$ where $k \neq 0$

$9 = \quad 3\sqrt{36}$

$k = \frac{1}{\quad}$

(b) From (a), $w = \frac{1}{2}u\sqrt{v}$

$w = \frac{1}{2}(\quad)\sqrt{81}$

$=$

5. Partial Variation

A quantity z may be composed of several parts. One of the parts varies as another quantity x and a second part varies as a third quantity y and so on. We call this kind of variation a partial variation and the equation involves a sum of several parts.

For example, if z varies partly directly as x and partly directly as y, then

$$z = k_1 x + k_2 y$$

where k_1 and k_2 are variation constants.

If z is partly constant and partly varies directly as x, then

$$z = k_1 + k_2 x$$

where k_1 and k_2 are variation constants.

Example 7

The profit y of an item is partly constant and partly varies directly as the number of items sold x. When $x = 1$, the profit y is 15. When $x = 100$, $y = 213$. Find

(a) the relation between the number of items sold x and the profit y,

(b) the profit when the number of items sold is 5.

Solution

Since y is partly constant and partly varies directly as x,

$y = k_1 + k_2 x$

When $x = 1$, $y = 15$

$15 = k_1 + k_2(\quad)$

$15 = k_1 + k_2$ .... (1)

When $x = 100$, $y = 213$,

$= k_1 + k_2(\qquad)$

$= k_1 + \qquad k_2$ .... (2)

(2)-(1) $198 = \qquad k_2$

$k_2 =$

Sub. k_2 into (1), $15 = k_1 +$

$k_1 =$

Hence $y = 13 + \quad x$

When $x = 5$, $y = 13 + 2(\quad) = \quad$.

Example 8

The average cost per head of providing a school lunch is partly constant and partly varies inversely as the number of students taking the lunch. When 144 students take lunch, the average cost is \$17 per head. The average cost becomes \$16 when the number of students rises to 168.

(a) Find an equation connecting the average cost per head and the number of students taking lunch.

(b) How many students would be required to reduce the cost of lunch to $14 ?

(c) If 1000 students would take lunch, a rebate of $500 would be given to the school. Calculate the actual average cost per head.

Solution

. (a) $c = a + \frac{b}{x}$

where cost = average cost per head

x = number of students

$17 = a + \frac{b}{}$ ----- *(1)*

$16 = a + \frac{b}{}$ ----- *(2)*

$(1) - (2),\ 1 = b\left(\frac{1}{144} - \frac{1}{}\right)$

b =

a =

$\therefore c = 10 + \frac{}{x}$

(b)

$= 10 + \frac{1008}{x}$

number of students x =252

(c)

cost for 1000 $= 10 + \frac{}{1000}$

=$11.01

Taking rebate $500 into account, the actual average cost per head

= $11.01-$0.

= $