99-00 Individual	1	$\frac{170}{891}$	2	3	3	10	4	35	5	540
	6	190	7	$\frac{1}{3}$	8	428571	9	24	10	0

99-00 Group	1	–3	2	5	3	6	4	10	5	10
	6	60	7	0.93	8	421	9	12	10	0

Individual Events

I1 Let $x = 0.\dot{1}\dot{7} + 0.0\dot{1}\dot{7} + 0.00\dot{1}\dot{7} + \dots$, find the value of x.

$0.\dot{1}\dot{7} = \frac{17}{99}$; $0.0\dot{1}\dot{7} = \frac{17}{990}$; $0.00\dot{1}\dot{7} = \frac{17}{9900}$

$$x = \frac{17}{99} + \frac{17}{990} + \frac{17}{9900} + \cdots = \frac{17}{99}\left(1 + \frac{1}{10} + \frac{1}{100} + \cdots\right) = \frac{17}{99} \cdot \frac{10}{9} = \frac{170}{891}$$

I2 Solve the following equation:

$$\frac{1}{x+12} + \frac{1}{(x+1)(x+2)} + \frac{1}{(x+2)(x+3)} + \frac{1}{(x+3)(x+4)} + \cdots + \frac{1}{(x+10)(x+11)} + \frac{1}{(x+11)(x+12)} = \frac{1}{4}.$$

$$\frac{1}{x+12} + \left(\frac{1}{x+1} - \frac{1}{x+2}\right) + \left(\frac{1}{x+2} - \frac{1}{x+3}\right) + \left(\frac{1}{x+3} - \frac{1}{x+4}\right) \cdots + \left(\frac{1}{x+10} - \frac{1}{x+11}\right) + \left(\frac{1}{x+11} - \frac{1}{x+12}\right) = \frac{1}{4}$$

$\frac{1}{x+1} = \frac{1}{4} \Rightarrow x = 3$

I3 Using digits 0, 1, 2, and 5, how many 3-digit numbers can be formed, which are divisible by 5? (If no digit may be repeated.)

Possible numbers are: 105, 120, 125, 150, 205, 210, 215, 250, 510, 520.

Altogether 10 numbers.

I4 Figure 1 represents a 4×3 rectangular spiderweb. If a spider walks along the web from A to C and it always walks either due East or due North. Find the total number of possible paths

The numbers at each of the vertices of in the following figure show the number of possible ways.

So the total number of ways = 35

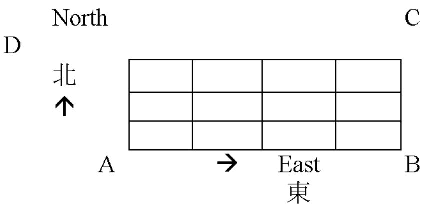
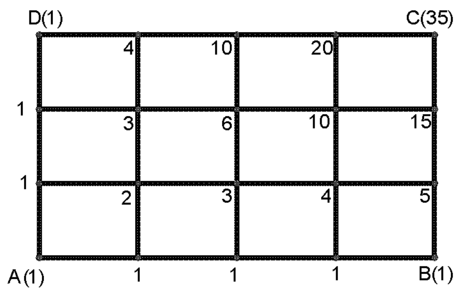


Figure 1 圖一



I5 In Figure 2, let $\angle A + \angle B + \angle C + \angle D + \angle E + \angle F + \angle G = x°$, find the value of x.

In the figure, let P, Q, R, S, T, U, V be as shown.

$\angle AVP + \angle BPQ + \angle CQR + \angle DRS + \angle EST + \angle FTU + \angle GUV = 360°$ (sum of ext. $\angle$ of polygon)

$\angle A = 180° - (\angle AVP + \angle BPQ)$ ($\angle$s sum of Δ)

$\angle B = 180° - (\angle BPQ + \angle CQR)$ ($\angle$s sum of Δ)

$\angle C = 180° - (\angle CQR + \angle DRS)$ ($\angle$s sum of Δ)

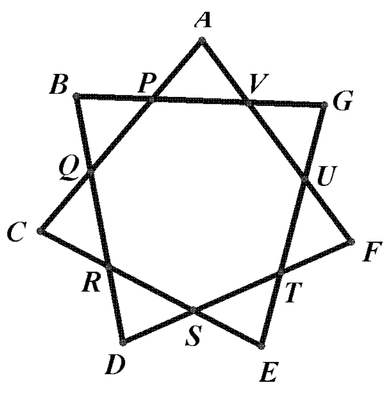
$\angle D = 180° - (\angle DRS + \angle EST)$ ($\angle$s sum of Δ)

$\angle E = 180° - (\angle EST + \angle FTU)$ ($\angle$s sum of Δ)

$\angle F = 180° - (\angle FTU + \angle GUV)$ ($\angle$s sum of Δ)

$\angle G = 180° - (\angle GUV + \angle AVP)$ ($\angle$s sum of Δ)

$\angle A + \angle B + \angle C + \angle D + \angle E + \angle F + \angle G = 180° \times 7 - 2 \times 360°$; $x = 540$



I6 Twenty straight lines were drawn on a white paper. Among them, no two or more straight lines are parallel; also no three or more than three straight lines are concurrent. What is the maximum number of intersections that these 20 lines can form?
2 lines give at most 1 intersection.
3 lines give at most 3 intersections.
4 lines give at most 6 intersections. (6 = 1 + 2 + 3)
………………………………………………………………………
20 lines give at most 1+2+3+…+19 intersections. $\frac{1+19}{2}\cdot 19 =$ 190 intersections

I7 In a family of 2 children, given that one of them is a girl, what is the probability of having another girl? (Assuming equal probabilities of boys and girls.)
Sample space = {(girl, boy), (girl, girl), (boy, girl)} and each outcome is equal probable.
$\therefore$ P(another child is also a girl) = $\frac{1}{3}$

I8 A particular 6-digit number has a unit-digit "1". Suppose this unit-digit "1" is moved to the place of hundred thousands, while the original ten thousand-digit, thousand-digit, hundred-digit, …are moved one digit place to the right. The value of the new 6-digit number is one-third of the value of the original 6-digit number. Find the original 6-digit number.
Let the original number be: $\overline{abcde1}$, and the new number be: $\overline{1abcde}$.
$3\times\overline{1abcde} = \overline{abcde1}$
3(100000 + 10000a + 1000b + 100c + 10d + e) = 100000a+10000b + 1000c + 100d + 10e + 1
Compare the unit digit: e = 7 with carry digit 2 to the tens digit
Compare the tens digit: d = 5 with carry digit 1 to the hundreds digit
Compare the hundreds digit: c = 8 with carry digit 2 to the thousands digit
Compare the thousands digit: b = 2 with no carry digit to the ten-thousands digit
Compare the ten-thousands digit: a = 4 with carry digit 1 to the hundred-thousands digit
The original number is 428571

I9 Find the value of $\frac{12\sin^2 48^\circ + 12\sin^2 42^\circ}{\sin 330^\circ \tan 135^\circ - \sin^2 48^\circ \sin^2 42^\circ \tan 180^\circ}$.

$$\frac{12\sin^2 48^\circ + 12\sin^2 42^\circ}{\sin 330^\circ \tan 135^\circ - \sin^2 48^\circ \sin^2 42^\circ \tan 180^\circ} = \frac{12\sin^2 48^\circ + 12\cos^2 48^\circ}{\left(-\frac{1}{2}\right)(-1) - \sin^2 48^\circ \sin^2 42^\circ \times 0} = \frac{12}{\frac{1}{2}} = 24$$

I10 Find the shortest distance between the line $3x - y - 4 = 0$ and the point (2, 2).

$$d = \left|\frac{Ax_0 + By_0 + C}{\sqrt{A^2 + B^2}}\right| = \left|\frac{3\times 2 - 2 - 4}{\sqrt{3^2 + (-1)^2}}\right| = 0$$

Method 2 Sub. (2, 2) into $3x - y - 4 = 0$, LHS = 3×2 – 2 – 4 = 0 = RHS
$\therefore$ (2, 2) lies on the line, the shortest distance = 0

Group Events

G1 If a is a root of $x^2 + 2x + 3 = 0$, find the value of $\frac{a^5 + 3a^4 + 3a^3 - a^2}{a^2 + 3}$.
Divide $(a^5 + 3a^4 + 3a^3 - a^2)$ by $(a^2 + 2a + 3)$, quotient = $a^3 + a^2 - 2a$, remainder = $6a$

$$\frac{a^5 + 3a^4 + 3a^3 - a^2}{a^2 + 3} = \frac{(a^2 + 2a + 3)(a^3 + a^2 - 2a) + 6a}{(a^2 + 2a + 3) - 2a} = \frac{6a}{-2a} = -3$$

G2 There are exactly n roots in the equation $(\cos^2\theta - 1)(2\cos^2\theta - 1) = 0$, where $0^\circ < \theta < 360^\circ$. Find the value of n.
$\cos\theta = 1, -1, \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$.
θ = 180°, 45°, 315°, 135°, 225°, n = 5

G3 Find the unit digit of 2004^{2006}.
$4^1 = 4$, $4^2 = 16$, $4^3 = 64$, $4^4 = 256$, …, so the unit digit of 2004^{2006} is 6.

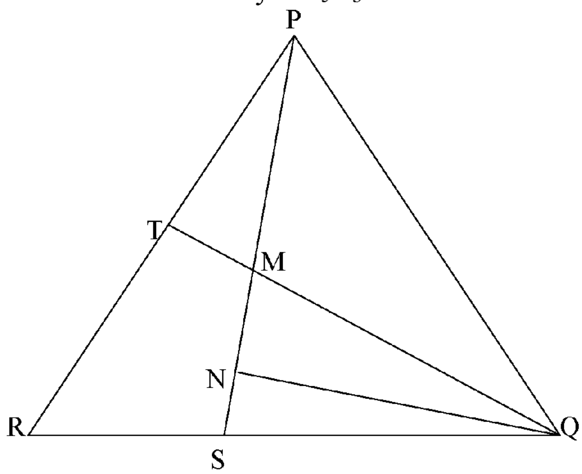
G4 Let $x = |y - m| + |y - 10| + |y - m - 10|$, where $0 < m < 10$ and $m \le y \le 10$. Find the minimum value of x.

$x = y - m + 10 - y + 10 - y + m = 20 - y \ge 20 - 10 = 10$, the minimum = 10

G5 There are 5 balls with labels A, B, C, D, E respectively and there are 5 pockets with labels A, B, C, D, E respectively. A ball is put into each pocket. Find the number of ways in which exactly 3 balls have labels that match the labels on the pockets.

First choose any 3 bags out of five bags. Put the balls according to their numbers. The remaining 2 balls must be put in the wrong order. The number of ways is ${}_5C_3 = 10$.

G6 In Figure 1, ΔPQR is an equilateral triangle, $PT = RS$; PS, QT meet at M; and QN is perpendicular to PS at N. Let $\angle QMN = x°$, find the value of x.



$PT = RS$ (given)

$\angle QPT = 60° = \angle PRS$ ($\angle$ of an equilateral Δ)

$PQ = PR$ (side of an equilateral Δ)

$\Delta PQT \cong \Delta RPS$ (SAS)

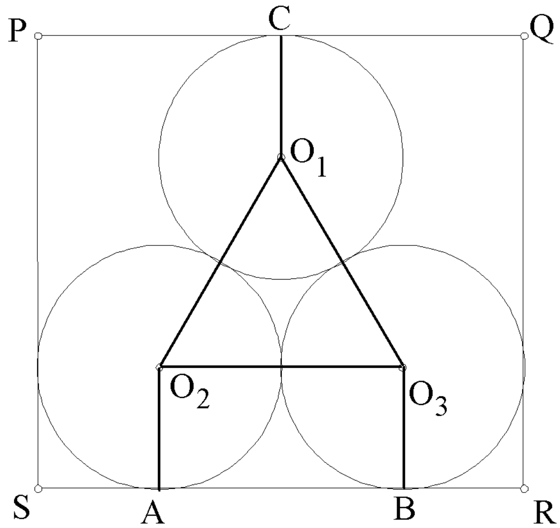
$\therefore \angle PTQ = \angle PSR$ (corr. $\angle$s $\cong \Delta$)

R, S, M, T are concyclic (ext. $\angle$ = int. opp. $\angle$)

$\angle QMN = x° = \angle TRS = 60°$ (ext. $\angle$, cyclic quad.)

$x = 60$

G7 In Figure 2, three equal circles are tangent to each other, and inscribed in rectangle $PQRS$, find the value of $\frac{QR}{SR}$. (Use $\sqrt{3} = 1.7$ and give the answer correct to 2 decimal places)



Let the radii of the circles be r.

Suppose the 3 circles touch the rectangle at A, B and C. Join O_1O_2, O_2O_3, O_1O_3, O_1C, O_2A, O_3B as shown. Then $O_1O_2 = O_2O_3 = O_1O_3 = 2r$

$O_1C = O_2A = O_3B = r$

$O_1O_2O_3$ is an equilateral Δ

$QR = O_1C + O_1O_2 \sin 60° + O_2A$

$= r + 2r \cdot \frac{\sqrt{3}}{2} + r = r(2 + \sqrt{3})$

$SR = 4r$, $\frac{QR}{SR} = \frac{r(2+\sqrt{3})}{4r} = \frac{2+1.7}{4} = \frac{37}{40} \approx 0.93$

G8 The sum of two positive integers is 29, find the minimum value of the sum of their squares.

Let the two numbers be a and b.

$a^2 + b^2 = (a - b)^2 + 2ab \ge 2ab$

The sum of squares is a minimum if their difference is a minimum.

$\because a$, and b are integers, the minimum of their difference is 1 ($a = 15$, $b = 14$)

$a^2 + b^2 = 15^2 + 14^2 = 225 + 196 = 421$

G9 Let $x = \sqrt{3+\sqrt{3}}$ and $y = \sqrt{3-\sqrt{3}}$, find the value of $x^2(1 + y^2) + y^2$.

$x^2(1 + y^2) + y^2 = (3+\sqrt{3})(1+3-\sqrt{3})+3-\sqrt{3} = (3+\sqrt{3})(4-\sqrt{3})+3-\sqrt{3} = 12+4\sqrt{3}-3\sqrt{3}-3+3-\sqrt{3} = 12$

G10 There are nine balls in a pocket, each one having an integer label from 1 to 9. A draws a ball randomly from the pocket and puts it back, then B draws a ball randomly from the same pocket. Let n be the unit digit of the sum of numbers on the two balls drawn by A and B, and P(n) be the probability of the occurrence of n. Find the value of n such that P(n) is the maximum.

Maximum = P(0) = P((1, 9), (2, 8), (3, 7), (4, 6), (5, 5), (6, 4), (7, 3), (8, 2), (9, 1)), $n = 0$

P(1) = P((2,9), (3,8), (4,7), (5,6), (6,5), (7,4), (8,3), (9,2))
P(2) = P((1,1), (3,9), (4,8), (5,7), (6,6), (7,5), (8,4), (9,3))
P(3) = P((1,2), (2,1), (4,9), (5,8), (6,7), (7,6), (8,5), (9,4))
P(4) = P((1,3), (2,2), (3,1), (5,9), (6,8), (7,7), (8,6), (9,5))
P(5) = P((1,4), (2,3), (3,2), (4,1), (6,9), (7,8), (8,7), (9,6))
P(6) = P((1,5), (2,4), (3,3), (4,2), (5,1), (7,9), (8,8), (9,7))
P(7) = P((1,6), (2,5), (3,4), (4,3), (5,2), (6,1), (8,9), (9,8))
P(8) = P((1,7), (2,6), (3,5), (4,4), (5,3), (6,2), (7,1), (9,9))
P(9) = P((1,8), (2,7), (3,6), (4,5), (5,4), (6,3), (7,2), (8,1))