08-09 Individual	**1**	$\frac{2300}{9801}$	**2**	2	**3**	60	**4**	4	**5**	$\frac{113}{4}$ (= 28.25)	**Spare**
	6	$\frac{2007}{2008}$	**7**	$9\sqrt{3}$	**8**	7	**9**	32	**10**	200	$-\frac{3}{2}$

08-09 Group	**1**	105	**2**	$2\sqrt{2}$	**3**	9	**4**	7	**5**	16	**Spare**
	6	$\frac{2}{3}$	**7**	–4	**8**	$\frac{21\sqrt{2}}{5}$	**9**	10	**10**	$\frac{8}{5}$ (= 1.6)	16

Individual Events

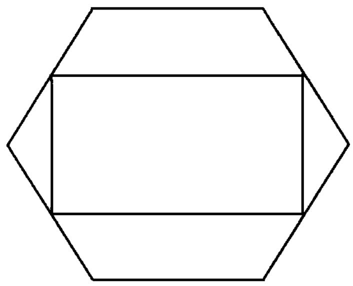
I1 Let $x = 0.\dot{2}\dot{3} + 0.00\dot{2}\dot{3} + 0.0000\dot{2}\dot{3} + 0.000000\dot{2}\dot{3} \ldots$, find the value of x.

Reference: HKMO 2000 Heat Individual Q1 Let $x = 0.\dot{1}\dot{7} + 0.0\dot{1}\dot{7} + 0.00\dot{1}\dot{7} + \ldots$

$$0.\dot{2}\dot{3} = \frac{23}{99};\quad 0.00\dot{2}\dot{3} = \frac{23}{9900};\quad 0.0000\dot{2}\dot{3} = \frac{23}{990000};\ \ldots$$

$$x = \frac{23}{99} + \frac{23}{9900} + \frac{23}{990000} + \cdots = \frac{23}{99}\left(1 + \frac{1}{100} + \frac{1}{10000} + \cdots\right) = \frac{23}{99}\cdot\frac{1}{1-\frac{1}{100}} = \frac{23}{99}\cdot\frac{100}{99} = \frac{2300}{9801}$$

I2 In Figure 1, a regular hexagon and a rectangle are given. The vertices of the rectangle are the midpoints of four sides of the hexagon. If the ratio of the area of the rectangle to the area of the hexagon is 1 : q, find the value of q.



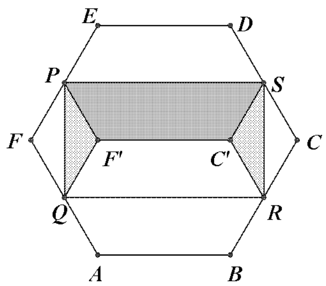
Let one side of the hexagon be $2a$, the height of the rectangle be x, the length be y.

$x = 2a \cos 30° = \sqrt{3}a$, $y = 2a \cos 60° + 2a = 3a$

Ratio of area $= \sqrt{3}a \cdot 3a : 6 \times \frac{1}{2}(2a)^2 \sin 60° = 3\sqrt{3} : 6\sqrt{3} = 1 : 2$

$q = 2$

The following method is provided by Mr. Jimmy Pang from Sai Kung Sung Tsun Catholic School (Secondary).



Let the hexagon be $ABCDEF$, the rectangle be $PQRS$ as shown.
Fold ΔFPQ along PQ to $\Delta F'PQ$. Fold ΔCSR along SR to $\Delta C'SR$. Fold $EDSP$ along PS to $F'C'SP$. Fold $ABRQ$ along QR to $F'C'RQ$. Then the folded figure covers the rectangle completely.

$\therefore$ Ratio of area = 1 : 2; $q = 2$

I3 Let $16 \sin^4 \theta° = 5 + 16 \cos^2 \theta°$ and $0 \le \theta \le 90$, find the value of θ.

$16 \sin^4 \theta° = 5 + 16 (1 - \sin^2 \theta°)$

$16 \sin^4 \theta° + 16 \sin^2 \theta° - 21 = 0$

$(4 \sin^2 \theta° - 3)(4 \sin^2 \theta° + 7) = 0$

$\sin^2 \theta° = \frac{3}{4}$ or $-\frac{7}{4}$ (rejected)

$\sin \theta° = \frac{\sqrt{3}}{2}$ or $-\frac{\sqrt{3}}{2}$ (rejected)

$\theta = 60$

I4 Let m be the number of positive factors of gcd(2008, 4518), where gcd(2008, 4518) is the greatest common divisor of 2008 and 4518. Find the value of m.

$2008 = 8\times251$; $4518 = 2\times9\times251$

gcd $= 2\times251 = 502$

The positive factors are 1, 2, 251, 502; $m = 4$

I5 Given that $x^2 + (y-3)^2 = 7$, where x and y are real numbers. If the maximum value of $5y + x^2$ is k, find the value of k.

$x^2 = 7 - (y-3)^2$; sub. into $5y + x^2 = 5y + 7 - (y-3)^2$

$= 5y + 7 - y^2 + 6y - 9$

$= -y^2 + 11y - 2 = -(y^2 - 11y + 5.5^2 - 5.5^2) - 2$

$= -(y - 5.5)^2 + 30.25 - 2 = = -(y - 5.5)^2 + 28.25$

Maximum value = k = 28.25

Method 2 $z = 5y + x^2 = -y^2 + 11y - 2$

$y^2 - 11y + (z + 2) = 0$

Discriminant $\Delta \ge 0$ for all real value of y.

$\therefore (-11)^2 - 4(1)(z+2) \ge 0 \Rightarrow 121 - 4z - 8 \ge 0$

$z \le \frac{113}{4}$; $k = \frac{113}{4}$

I6 Let $f_1(x) = \frac{1}{1-x}$ and $f_n(x) = f_1(f_{n-1}(x))$, where $n = 2, 3, 4, \ldots$. Find the value of $f_{2009}(2008)$.

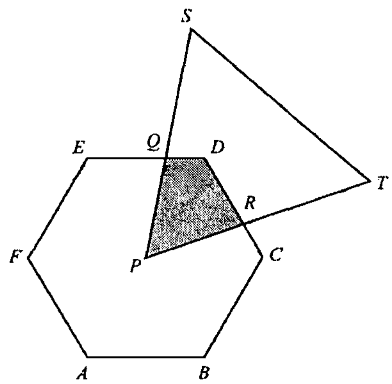
Reference: HKMO Heat 1996-97 Group Q2: If $f(x) = \frac{2x}{x+2}$ and $x_1 = 1$, $x_n = f(x_{n-1})$, find x_{99}.

$$f_2(x) = f_1(f_1(x)) = f_1\left(\frac{1}{1-x}\right) = \frac{1}{1-\frac{1}{1-x}} = \frac{1-x}{1-x-1} = \frac{x-1}{x}$$

$$f_3(x) = f_1(f_2(x)) = f_1\left(\frac{x-1}{x}\right) = \frac{1}{1-\left(1-\frac{1}{x}\right)} = x, \therefore f_{3n}(x) = x \text{ for all positive integer } n.$$

$$f_{2009}(2008) = f_2(f_{2007}(2008)) = f_2(f_{3(669)}(2008)) = f_2(2008) = \frac{2008-1}{2008} = \frac{2007}{2008}$$

I7 In Figure 2, $ABCDEF$ is a regular hexagon centred at the point P. ΔPST is an equilateral triangle. It is given that AB = 6 cm, QD = 2 cm and PT = 12 cm. If the area of the common part of the hexagon and triangle is c cm^2, find the value of c.



Join PD and CP. ΔCDP is an equilateral triangle, side = 6 cm

$\angle QPR + \angle QDR = 60° + 120° = 180°$

$\therefore DQPR$ is a cyclic quadrilateral (opp. sides supp.)

$\angle PQD = \angle PRC$ (ext. $\angle$, cyclic quad.)

$\angle DPQ = 60° - \angle DPR = \angle CPR$

$CP = DP$ sides of an equilateral triangle

$\therefore \Delta CPR \cong \Delta DPQ$ (AAS)

Shaded area = area of ΔPDQ + area of ΔPDR

= area of ΔCPR + area of ΔPDR = area ΔCPD

$= \frac{1}{2} \cdot 6^2 \cdot \sin 60° = 9\sqrt{3}$ cm^2; $c = 9\sqrt{3}$

I8 Find the unit digit of 7^{2009}.

Reference: HKMO 2006 heat individual Q9: Given that the units digit of 7^{2006} is C, ...

$7^1 = 7$, $7^2 \equiv 9$ (mod 10), $7^3 \equiv 3$ (mod 10), $7^4 \equiv 1$ (mode 10)

$7^{2009} = (7^4)^{502} \times 7 \equiv 7$ mod 10

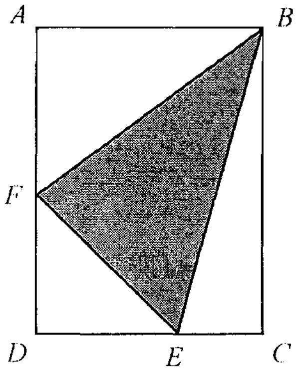
I9 Given that a and b are integers. Let $a - 7b = 2$ and $\log_{2b} a = 2$, find the value of $a \times b$.

$(2b)^2 = a \Rightarrow 4b^2 = 7b + 2 \Rightarrow 4b^2 - 7b - 2 = 0 \Rightarrow (b-2)(4b+1) = 0$

$b = 2$ or $-\frac{1}{4}$ (rejected)

$a = 7b + 2 = 16$; $a \times b = 32$

110 In Figure 3, $ABCD$ is a rectangle. Points E and F lie on CD and AD respectively, such that $AF = 8$ cm and $EC = 5$ cm. Given that the area of the shaded region is 80 cm^2. Let the area of the rectangle $ABCD$ be g cm^2, find the value of g.



Let $DF = x$ cm, $DE = y$ cm

$$g=\frac{1}{2}xy+\frac{1}{2}(8+x)\cdot 5+\frac{1}{2}(y+5)\cdot 8+80$$

$$=\frac{1}{2}(xy+5x+8y+40+200)$$

$$=\frac{1}{2}[(x+8)(y+5)+200]=\frac{1}{2}g+100\Rightarrow g=200$$

IS Given that a is a negative real number. If $\frac{1}{a+\frac{1}{a+2}}=2$, find the value of a.

$$1=2a+\frac{2}{a+2}\Rightarrow 1-2a=\frac{2}{a+2}\Rightarrow 2-3a-2a^2=2\Rightarrow a=-\frac{3}{2}$$

Group Events

G1 If a is a positive integer and $\frac{1}{a(a+1)}+\frac{1}{(a+1)(a+2)}+\cdots+\frac{1}{2008\times 2009}=\frac{272}{30135}$, find the value of a.

$$\left(\frac{1}{a}-\frac{1}{a+1}\right)+\left(\frac{1}{a+1}-\frac{1}{a+2}\right)+\cdots+\left(\frac{1}{2008}-\frac{1}{2009}\right)=\frac{272}{30135}$$

$$\frac{1}{a}-\frac{1}{2009}=\frac{272}{30135}\Rightarrow\frac{1}{a}=\frac{272}{2009\times 15}+\frac{1}{2009}=\frac{272+15}{2009\times 15}=\frac{287}{3\times 5\times 7^2\times 41}=\frac{1}{15\times 7};\ a=105$$

G2 Let $x=1+\sqrt{2}$, find the value of $x^5-2x^4+3x^3-4x^2-10x-6$.

Reference HKMO 2007 Heat Group Q3 … $a^2-3a+1=0$. If $A=\frac{2a^5-5a^4+2a^3-8a^2+7a}{3a^2+3}$, …

$x-1=\sqrt{2}$, $(x-1)^2=2\Rightarrow x^2-2x-1=0$

By division, $x^5-2x^4+3x^3-4x^2-10x-6=(x^2-2x-1)(x^3+4x+4)+2x-2=2x-2$

$$=2(1+\sqrt{2})-2=2\sqrt{2}$$

Method 2 Divide $x^5-2x^4+3x^3-4x^2-10x-6$ by $(x-1)$ successively:

1	1	–2	3	–4	–10	–6
		1	–1	2	–2	–12
1	1	–1	2	–2	–12	–18
		1	0	2	0	
1	1	0	2	0	–12	
		1	1	3		
1	1	1	3	3		
		1	2			
1	1	2	5			
		1				
	1	3				

Let $y=x-1$, the expression in terms of y becomes

$y^5+3y^4+5y^3+3y^2-12y-18$

$$=\sqrt{2}^5+3(\sqrt{2})^4+5(\sqrt{2})^3+3(\sqrt{2})^2-12\sqrt{2}-18$$

$$=4\sqrt{2}+12+10\sqrt{2}+6-12\sqrt{2}-18$$

$$=2\sqrt{2}$$

G3 Given that p and q are integers. If $\frac{2}{p}+\frac{1}{q}=1$, find the maximum value of $p\times q$.

Reference HKMO 2008 Heat Individual Q3 … $\frac{1}{x}+\frac{1}{y}=\frac{1}{15}$. If $35 < y_0 < 50$ and $x_0 + y_0 = z_0$ …

$2q + p = pq \Rightarrow pq - p - (2q - 2) = 2 \Rightarrow (p - 2)(q - 1) = 2$

$(p - 2, q - 1) = (1, 2), (2, 1), (-1, -2), (-2, -1)$

$(p, q) = (3, 3), (4, 2), (1, -1), (0, 0) \Rightarrow$ maximum $p\times q = 3\times 3 = 9$

G4 Given that $0 \le x \le 180$. If the equation $\cos 7x° = \cos 5x°$ has r distinct roots, find the value of r.

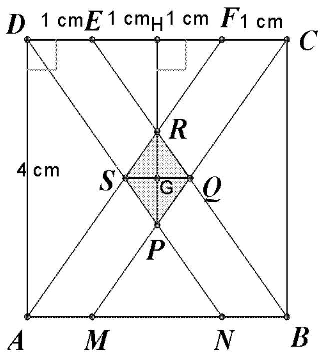
$7x = 5x + 360n$	or	$7x = 360 - 5x + 360n$
$x = 180n$	or	$x = 30(n + 1)$
$n = 0, x = 0$	or	30
$n = 1, x = 180$	or	60
$n = 2$		$x = 90$
$n = 3$		$x = 120$
$n = 4$		$x = 150$
$r = 7$		

G5 Let x, y and z be positive integers and satisfy $\sqrt{z-\sqrt{28}}=\sqrt{x}-\sqrt{y}$. Find the value of $x + y + z$.

$z-\sqrt{28}=\left(\sqrt{x}-\sqrt{y}\right)^2 \Rightarrow z-2\sqrt{7}=x+y+2\sqrt{xy} \Rightarrow x+y=z;\ xy=7$

$\therefore x = 7, y = 1; z = 1 + 7 = 8; x + y + z = 16$

G6 In Figure 1, $ABCD$ is a square and $AM = NB = DE = FC = 1$ cm and $MN = 2$ cm. Let the area of quadrilateral $PQRS$ be c cm^2, find the value of c.



$DF = 3$ cm, $AD = CD = 4$ cm, $AF=\sqrt{3^2+4^2}$ cm $= 5$ cm $= DN$

$\Delta ADF \cong \Delta NFD$ (SAS) $\Rightarrow \angle AFD = \angle NDF$ (corr. $\angle$s $\cong \Delta$'s)

$\therefore DS = FS$ (sides opp. eq. $\angle$s)

But $\Delta DSF \cong \Delta NSA$ (ASA) $\Rightarrow AS = SF = \frac{1}{2}AF = \frac{5}{2}$ cm

Let H be the mid point of EF. $EH = HF = 1$ cm

Suppose SQ intersects PR at G.

It is easy to show that $\Delta ADF \sim \Delta RHF \sim \Delta RGS$

$\frac{FR}{AF}=\frac{FH}{AD} \Rightarrow \frac{FR}{5\text{ cm}}=\frac{1}{3} \Rightarrow FR=\frac{5}{3}$ cm (ratio of sides, ~Δ's)

$SR = SF - RF = \frac{5}{2}\text{ cm} - \frac{5}{3}\text{ cm} = \frac{5}{6}$ cm

$\frac{\text{Area }\Delta\text{RGS}}{\text{Area }\Delta\text{ADF}}=\left(\frac{SR}{AF}\right)^2 \Rightarrow \frac{\text{Area }\Delta\text{RGS}}{\frac{1}{2}\cdot 3\cdot 4\text{cm}^2}=\left(\frac{\frac{5}{6}}{5}\right)^2 \Rightarrow \text{Area }\Delta RSG=\frac{1}{6}\text{ cm}^2$

Area of $PQRS = 4$ area of $\Delta RSG = \frac{2}{3}$ cm^2, $c = \frac{2}{3}$

Method 2 Let $\angle RFH = \theta = \angle PCH = \angle CMB$ (corr. $\angle$s and alt. $\angle$s, //-lines)

$\tan\theta=\frac{HR}{1\text{cm}}=\frac{HP}{2\text{ cm}}=\frac{4}{3} \Rightarrow HR=\frac{4}{3}$ cm; $HP=\frac{8}{3}$ cm

$RP = HP - HR = \left(\frac{8}{3}-\frac{4}{3}\right)$ cm $= \frac{4}{3}$ cm

Further, $FCQS$ is a //-gram. $\therefore SQ = FC = 1$ cm (opp. sides, //-gram)

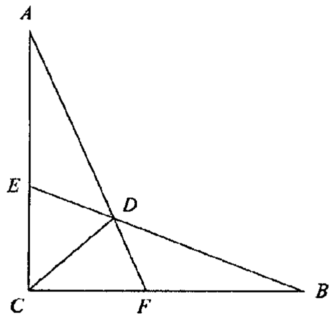
Area of $PQRS = \frac{1}{2}PR\cdot SQ = \frac{1}{2}\cdot\frac{4}{3}\times 1\text{ cm}^2$; $c = \frac{2}{3}$

G7 Given that x is a real number and satisfies $2^{2x+8} + 1 = 32\times 2^x$. Find the value of x.

Let $y = 2^x$, $y^2 = 2^{2x}$, the equation becomes: $2^8\cdot y^2 + 1 = 32y$

$256y^2 - 32y + 1 = 0 \Rightarrow (16y - 1)^2 = 0 \Rightarrow y = \frac{1}{16} \Rightarrow 2^x = 2^{-4}$; $x = -4$

G8 In Figure 2, $\angle ABC$ is a right angle, $AC = BC = 14$ cm and $CE = CF = 6$ cm. If $CD = d$ cm, find the value of d.



$\angle ACD = \angle BCD = 45°$

We find the area of $ACBD$ in two different ways.

$S_{ACBD} = S_{ACD} + S_{BCD} = S_{ACF} + S_{BCE} - S_{CDE} - S_{CDF}$

$2\times\frac{1}{2}\cdot 14d\sin 45° = 2\times\frac{1}{2}\cdot 6\times 14 - 2\times 2\times\frac{1}{2}\cdot 6x\sin 45°$

$20d\cdot\frac{1}{\sqrt{2}} = 84 \Rightarrow d = \frac{21\sqrt{2}}{5}$

Method 2 Set up a coordinate system with BC as x-axis, CA as y-axis, and C as the origin.

Equation of AF in intercept form: $\frac{x}{6} + \frac{y}{14} = 1$ … (1)

Equation of CD is $y = x$ … (2)

Sub. (2) into (1): $\frac{x}{6} + \frac{x}{14} = 1 \Rightarrow x = \frac{21}{5}$ $\therefore d = \frac{21}{5}\sin 45° = \frac{21\sqrt{2}}{5}$

G9 If there are 6 different values of real number x that satisfies $||x^2 - 6x - 16| - 10| = f$, find the value of f.

$|x^2 - 6x - 16| - 10 = \pm f$

$|x^2 - 6x - 16| = 10 \pm f$

$x^2 - 6x - 16 = 10 + f$, $x^2 - 6x - 16 = 10 - f$, $x^2 - 6x - 16 = -10 + f$, $x^2 - 6x - 16 = -10 - f$

$x^2 - 6x - 26 - f = 0$, $x^2 - 6x - 26 + f = 0$, $x^2 - 6x - 6 - f = 0$, $x^2 - 6x - 6 + f = 0$

Each equation has 2 roots. $\therefore$ There are 8 roots. But the question assumes 6 repeated roots.

Suppose α is a common root of any two equations $x^2 - 6x + k_1 = 0$ and $x^2 - 6x + k_2 = 0$

Case 1 $-26 - f = -26 + f \Rightarrow f = 0$

Case 2 $-26 - f = -6 - f \Rightarrow$ no solution for f

Case 3 $-26 - f = -6 + f \Rightarrow f = -10$ (rejected, because $||x^2 - 6x - 16| - 10| = f$ gives no real root.

Case 4 $-26 + f = -6 + f \Rightarrow$ no solution for f

Case 5 $-26 + f = -6 - f \Rightarrow f = 10$

Case 6 $-6 - f = -6 + f \Rightarrow f = 0$

When $f = 0$, the equations are $x^2 - 6x - 26 = 0$ and $x^2 - 6x - 6 = 0$, which have only 4 roots rejected

When $f = 10$, the equations are $x^2 - 6x - 36 = 0$, $x^2 - 6x - 16 = 0$ and $x^2 - 6x + 4 = 0$, which have 6 roots (accepted).

Method 2

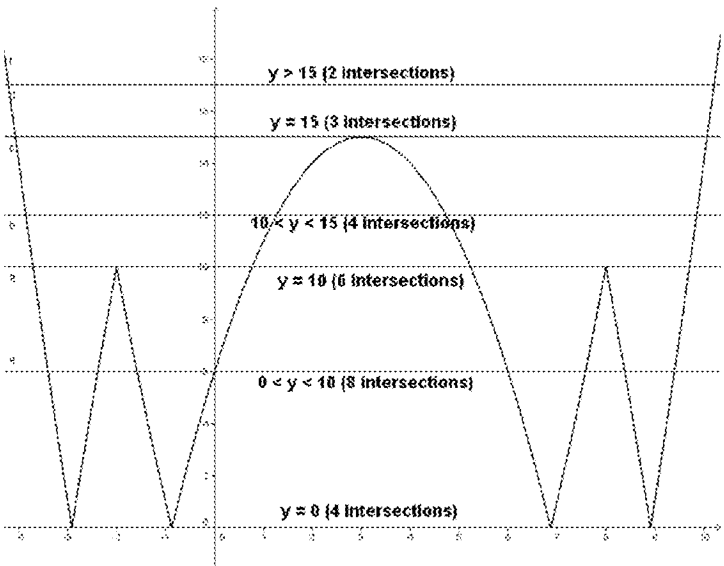
The following method is provided by Mr. Jimmy Pang from Sai Kung Sung Tsun Catholic School (Secondary).

First sketch the graph $y = ||x^2 - 6x - 16| - 10|$.

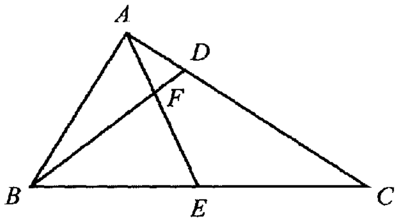
From the graph, draw different horizontal lines.

The line $y = 10$ cuts $y = ||x^2 - 6x - 16| - 10|$ at 6 different points.

$\therefore f = 6$



G10 In Figure 3, ABC is a triangle, E is the midpoint of BC, F is a point on AE where $AE = 3AF$. The extension segment of BF meets AC at D. Given that the area of ΔABC is 48 cm^2. Let the area of ΔAFD be g cm^2, find the value of g.

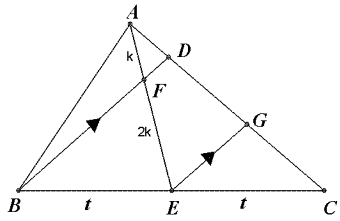


From E, draw a line EG // BD which cuts AC at G.

$AE = 3AF \Rightarrow AF : FE = 1 : 2$; let $AE = k$, $FE = 2k$

E is the midpoint of $BC \Rightarrow BE = EC = t$

$S_{ABE} = S_{ACE} = \frac{1}{2}\cdot 48$ cm^2 = 24 cm^2 (same base, same $\therefore$ height)



$AD : DG = AF : FE = 1 : 2$ (theorem of equal ratio)

$DG : GC = BE : EC = 1 : 1$ (theorem of equal ratio)

$\therefore AD : DG : GC = 1 : 2 : 2$

$S_{AEG} : S_{CEG} = 3 : 2$ (same height, ratio of base = 3 : 2)

$$S_{AEG} = 24\times\frac{3}{2+3}\text{cm}^2 = \frac{72}{5}\text{ cm}^2$$

$$\Delta ADF \sim \Delta AGE \Rightarrow S_{ADF} = \frac{1}{9}S_{AEG} = \frac{1}{9}\cdot\frac{72}{5}\text{cm}^2 = \frac{8}{5}\text{cm}^2;\ g = \frac{8}{5}$$

GS Let R be the remainder of 588^{2009} divided by 97. Find the value of R.

Reference: HKMO 2008 final individual spare Q4: … the remainder of 588^{2008} divided by 97 …

$588^{2009} = (97\times 6 + 6)^{2009}$

$= (97\times 6)^{2009} + {}_{2009}C_1\cdot(97\times 6)^{2009}\times 6 + \ldots + 6^{2009} = 97m + 6^{2009}$, where m is an integer.

Note that $2^5\times 3 = 96 = 97 - 1 \equiv -1 \pmod{97}$; $2\times 3^5 = 486 = 97\times 5 + 1 \equiv 1 \pmod{97}$;

$\therefore 6^6 = (2^5\times 3)\times(2\times 3^5) \equiv -1 \pmod{97}$

$6^{2009} = (6^6)^{334}\times 6^5 \equiv (-1)^{334}\times 2^4\times(2\times 3^5) \equiv 16 \pmod{97}$; $d = 16$

HKMO 2009 Geometrical Construction Sample Paper solution

1. 在下列三角形中，試作出點使它與該三角形各邊的距離相等。

First, we find the **locus** of a point D which is equidistance to a given angle $\angle BAC$. (Figure 1)

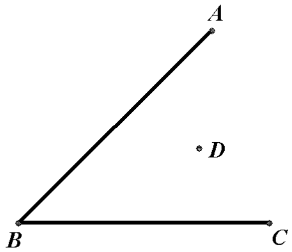


Figure 1

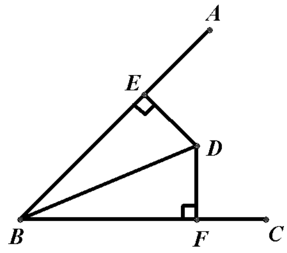


Figure 2

From D, let E and F be the feet of perpendiculars onto AB and BC respectively. (Figure 2)

Join BD. Then $BD = BD$ (common side)

$\angle BED = \angle BFD = 90°$ (By construction)

$DE = DF$ (given that D is equidistance to AB and BC)

$\therefore \Delta BDE \cong \Delta BDF$ (RHS)

$\angle DBE = \angle DBF$ (corr. $\angle$s $\cong \Delta$'s)

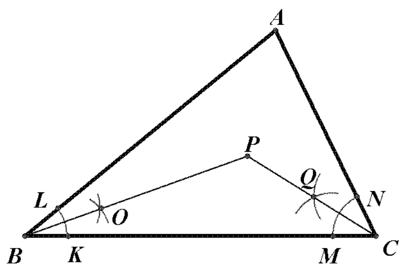
$\therefore$ D lies on the **angle bisector** of $\angle ABC$.

$\therefore$ In ΔABC, if P is equidistance to ABC, then P must lie on the **intersection of the three angle bisectors**. (i.e. the incentre of ΔABC.)

$\because$ The three angle bisectors must concurrent at one point

$\therefore$ We need to find the intersection of any two angle bisectors.

The construction is as follows:



With fixed radius centres at B and C, draw two circular arcs LK and MN respectively.

Using the same radius, centres at K and L, draw arcs intersect at O.

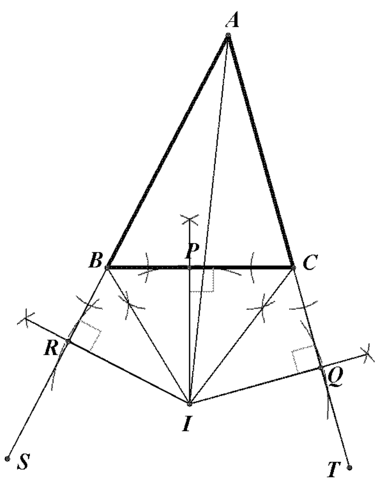
Using the same radius, centres at M and N, draw arcs intersect at Q.

Then BO is the bisector of $\angle ABC$.

Then CQ is the bisector of $\angle ACB$.

P is the intersection of BO and CQ.

P is the required point.



Note that the 3 **ex-centres** are equidistance from AB, BC, CA. I shall draw one ex-centre as demonstration.

(1) From AB produced, draw the exterior $\angle$ bisector BI.

(2) From AC produced, draw the exterior $\angle$ bisector CI.

(3) BI and CI intersect at I.

(4) From I, drop the perpendicular lines IP, IQ, IR to BC, AC and AB respectively. Join AI.

(5) $\Delta IBP \cong \Delta IBR$ (AAS); $\Delta ICP \cong \Delta ICQ$ (AAS)

$\therefore$ $IP = IQ = IR$ (corr. sides, $\cong \Delta$'s)

$\therefore$ I is equidistance from AB, BC and AC.

2. 已知一直線 L，及兩點 P、Q 位於 L 的同一方。試在 L 上作一點 T 使得 PT 及 QT 的長度之和最小。(Figure 1)

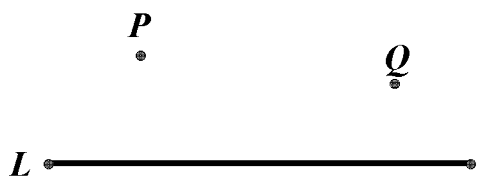


Figure 1

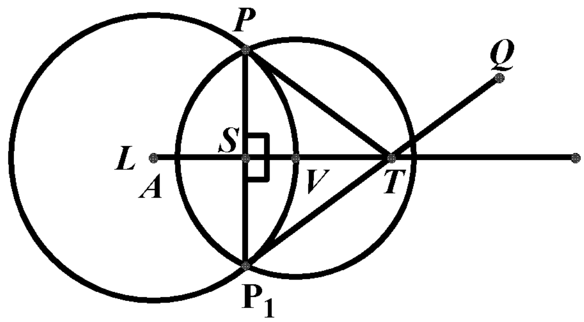


Figure 2

Let A be one of the end point of L nearer to P.

Using A as centre, AP as radius to draw a circle, which intersects L at V.

Using V as centre, VP as radius to draw another circle. These two circles intersect at P, P_1.

Join PP_1, which intersects L at S.

$PV = P_1V$	(same radii)
$AV = AV$	(common)
$AP = AP_1$	(same radii)
$\Delta APV \cong AP_1V$	(S.S.S.)
$\therefore \angle PAV = \angle P_1AV$	(corr. $\angle$s $\cong$ Δ's)
$AS = AS$	(common sides)
$AP = AP_1$	(same radii)
$\Delta APS \cong AP_1S$	(S.A.S.)
$\therefore \angle ASP = \angle ASP_1$	(corr. $\angle$s $\cong$ Δ's)
$= 90°$	(adj. $\angle$s on st. line)
$PS = SP_1$	(corr. sides $\cong$ Δ's)

Join P_1Q, which intersects L at T.

$ST = ST$	(common sides)
$\angle PST = 90° = \angle P_1ST$	(by construction)
$PS = PS$	(by construction)
$\therefore \Delta PST \cong \Delta P_1ST$	(S.A.S.)
$PT = P_1T$	(corr. sides $\cong$ Δ's)

$PT + QT = P_1T + QT$

We know that $P_1T + QT$ is a minimum when P_1, T, Q are collinear.

$\therefore$ T is the required point.

3. 試繪畫一固定圓的切線，且該切線通過一固定點 P。(註：P 在該圓形外。)

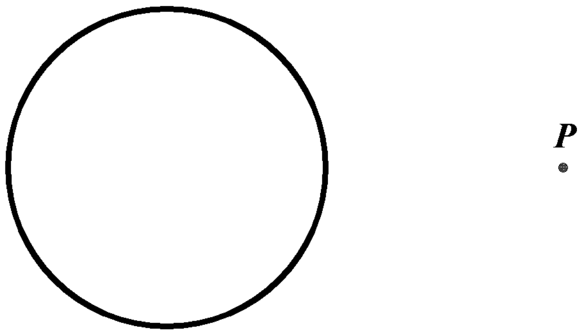


Figure 1

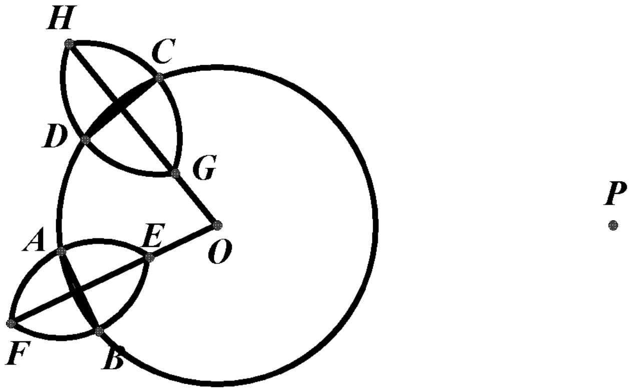


Figure 2

First, we locate the centre of the circle. (Figure 2)
Let AB and CD be two non-parallel chords.
Using A as centre, AB as radius to draw an arc. Using B as centre, BA as radius to draw another arc. The two arcs intersect at E, F. Join EF, which is the perpendicular bisector of AB.
Using C as centre, CD as radius to draw an arc. Using D as centre, DC as radius to draw another arc. The two arcs intersect at HG. Join HG, which is the perpendicular bisector of CD.
The two perpendicular bisectors EF and HG intersect at O, which is the centre of the circle.
Using O as the centre and a fixed distance (> half of OP) as radius to draw an arc.
Using P as the centre and the same radius to draw another arc.
The two arcs intersect at L and M. Join LM. This is the perpendicular bisector of OP, which intersect OP at K.
11 1 56 0 Using K as centre, KP as radius to draw a semi-circle, which intersects the given circle at N.
Join ON, NP.
$\angle ONP = 90°$ ($\angle$ in semi-circle)
PN is the required tangent.
(converse, tangent $\perp$ radius)

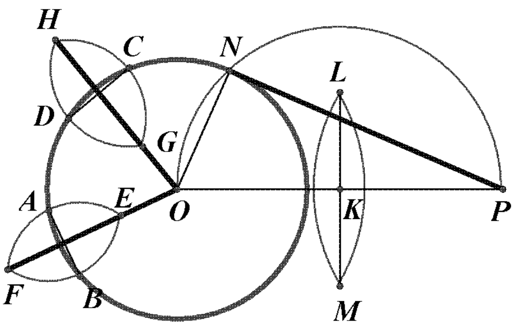
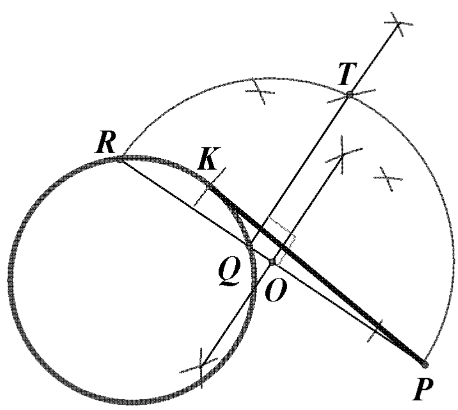


Figure 3

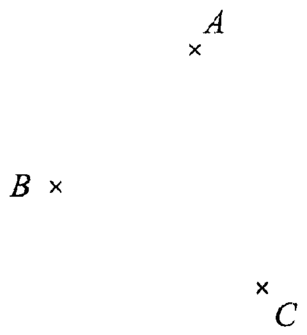
Method 2

(1) From P, draw a line segment cutting the circle at R and Q.
(2) Draw the perpendicular bisector of PR, which intersects PR at O.
(3) Use O as centre, $OP = OR$ as radius to draw a semi-circle PTR.
(4) From Q (the point between PR), draw a line QT perpendicular to PR, cutting the semi-circle at T.
$PQ \cdot QR = PT^2$ (intersecting chords theorem)
(5) Use P as centre, PT as radius to draw an arc, cutting the given circle at K.
(6) Join PK.
By (4), $PQ \cdot QR = PT^2 = PK^2$
$\therefore$ PK is the tangent from external point.
(converse, intersecting chords theorem)



Geometrical construction

1. In Figure 1, A, B and C are three fixed points. Construct a circle passing through the given three points.



The 3 perpendicular bisectors of AB, BC, AC are concurrent at the circumcentre O. It is sufficient to locate the circumcentre by any two perpendicular bisectors.

Step 1 Join AB, AC.

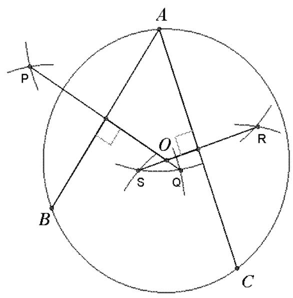
Step 2 Use A as centre and a fixed radius bigger than $\frac{1}{2}AB$, $\frac{1}{2}AC$ to draw an arc.

Step 3 Use B as centre and the same radius as step 2 to draw an arc.

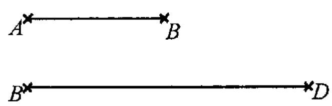
Step 4 Use C as centre and the same radius as step 2 to draw an arc.

Step 5 The arcs in step 2 and step 3 intersect at P, Q. The arcs in step 2 and step 4 intersect at R, S. Join PQ and RS. PQ and RS intersect at O.

Step 6 It can be proved that $OA = OB = OC$. Use O as centre and OA as radius, draw a circle to pass through A, B and C.



2. In Figure 2, AB is the base of a triangle ABC, and the length of BD is the sum of the lengths of BC and CA. Given that $\angle ABC = 60°$, construct the triangle ABC.



(1) Use A as centre, AB as radius to draw an arc, use B as centre, BA as radius to draw another arc. The two arcs intersect at P. ABP is an equilateral triangle. $\angle ABP = 60°$

(2) With B as centre, BD as radius, draw an arc, cutting BP produced at D.

(3) Join AD. Use A as centre and a fixed radius bigger than $\frac{1}{2}AD$ to draw an arc. Use D as centre and the same radius to draw an arc. The two arcs intersect at Q and R.

(4) Join RQ. RQ produced cuts BD at C. Join AC.

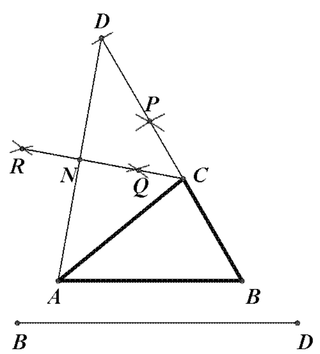
RQ is the perpendicular bisector of AD.

$\Delta ANC \cong \Delta DNC$ (SAS)

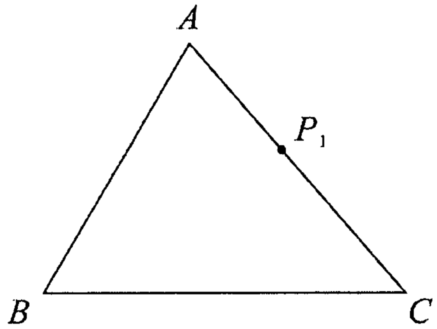
$\therefore AC = DC$ (corr. sides, $\cong \Delta$'s)

$AC + CB = BC + CD$

ΔABC is the required triangle.



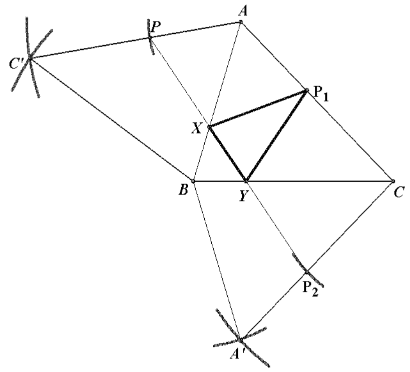
3. In Figure 3, ΔABC is an acute angle triangle. P_1 is a point on AC. Construct a triangle P_1XY such that X is a point on AB, Y is a point on BC and the perimeter of ΔP_1XY is the least.



(1) Use A as centre, AC as radius to draw an arc, use B as centre, BC as radius to draw another arc. These two arcs intersect at C'. Then $\Delta ABC' \cong \Delta ABC$ (SSS)

(2) Use C as centre, CA as radius to draw an arc, use B as centre, BA as radius to draw another arc. These two arcs intersect at A'. Then $\Delta A'BC \cong \Delta ABC$ (SSS)

(3) Use A as centre, AP_1 as radius to draw an arc, which cuts AC' at P. Use C as centre, CP_1 as radius to draw an arc, which cuts CA' at P_2.



(4) Join PP_2, which cuts AB at X and BC at Y. Join P_1X, XY, YP_2.

$\Delta AP_1X \cong \Delta APX$ (SAS)

$\therefore P_1X = PX$ (corr. sides $\cong \Delta$'s)

$\Delta CP_1Y \cong \Delta CP_2Y$ (SAS)

$\therefore CP_2 = CP_1$ (corr. sides $\cong \Delta$'s)

Perimeter of $\Delta P_1XY = PX + XY + YP_2$

$PX + XY + YP_2$ is the minimum when P, X, Y, P_2 are collinear.

ΔP_1XY is the required triangle.