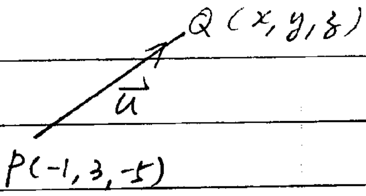
Suggested Solution



1. (a) Let Q be (x, y, z)

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= (x+1,\ y-3,\ z+5)$

Since $\vec{u}$ has the same length & in the same direction as $\vec{v}$,

we have $(x+1,\ y-3,\ z+5) = (6, 7, -3)$

$\therefore\ x+1 = 6\ ;\ y-3 = 7\ ;\ z+5 = -3$

$\therefore\ x = 5\ ;\ y = 10\ ;\ z = -8$

$\therefore$ Q has coordinates $(5, 10, -8)$

(b) Use similar approach as in 1(a), but this time $\vec{u}$ is in the opposite direction as $\vec{v}$, we have

$(x+1,\ y-3,\ z+5) = (-6, -7, 3)$

$\therefore\ x+1 = -6\ ;\ y-3 = -7\ ;\ z+5 = 3$

$\therefore\ x = -7\ ;\ y = -4\ ;\ z = -2$

$\therefore$ Q has coordinates $(-7, -4, -2)$

2. Given $\vec{u} = (-3, 1, 2)$, $\vec{v} = (4, 0, -8)$, $\vec{w} = (6, -1, -4)$. Let $\vec{x} = (a, b, c)$, Then,

$2\vec{u} - \vec{v} + \vec{x} = 7\vec{x} + \vec{w}$

$2(-3, 1, 2) - (4, 0, -8) + (a, b, c) = 7(a, b, c) + (6, -1, -4)$

$(-6, 2, 4) - (4, 0, -8) + (a, b, c) = (7a, 7b, 7c) + (6, -1, -4)$

$(-10+a,\ 2+b,\ 12+c) = (7a+6,\ 7b-1,\ 7c-4)$

$\therefore\ a-10 = 7a+6$

$b+2 = 7b-1$

$c+12 = 7c-4$

$\therefore\ \vec{x} = (-8/3,\ 1/2,\ 8/3)$

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3. (a) Let $\vec{u}=(a,b)$ be any non-zero vector, we have

$|\vec{u}| = \sqrt{a^2+b^2}$.

Let $\vec{v}$ be a vector formed by $\frac{\vec{u}}{|\vec{u}|}$, then

$\vec{v} = \frac{ai+bj}{\sqrt{a^2+b^2}} = \frac{a}{\sqrt{a^2+b^2}}i + \frac{b}{\sqrt{a^2+b^2}}j$

So $|\vec{v}| = \sqrt{\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2+\left(\frac{b}{\sqrt{a^2+b^2}}\right)^2}$

$= \sqrt{\frac{a^2+b^2}{a^2+b^2}}$

$= 1$

$\therefore$ $\vec{v}$ has a magnitude of 1 unit $\sim$ $\frac{\vec{u}}{|\vec{u}|}$ is a unit vector.

(b) $\vec{u}=(3,4)$, $\hat{u} = \left(\frac{3}{\sqrt{3^2+4^2}}, \frac{4}{\sqrt{3^2+4^2}}\right) = \left(\frac{3}{5}, \frac{4}{5}\right)$

(c) $\vec{u}=(-2,3,-6)$, $\hat{u} = \left(\frac{-2}{\sqrt{(-2)^2+3^2+(-6)^2}}, \frac{3}{\sqrt{(-2)^2+3^2+(-6)^2}}, \frac{-6}{\sqrt{(-2)^2+3^2+(-6)^2}}\right)$

opposite vector $V = (2,-3,6)$

$\therefore$ $\hat{v} = \left(\frac{+2}{7}, \frac{-3}{7}, \frac{+6}{7}\right)$

So a unit vector in the opposite direction as $\hat{u}$ is

$(2/7, -3/7, 6/7)$

4. Given $\vec{u}=(3,4)$, $\vec{v}=(5,-1)$, $\vec{w}=(7,1)$

(a) $\vec{u}\cdot(7\vec{v}+\vec{w})$

$= (3,4)\cdot[7(5,-1)+(7,1)]$

$= (3,4)\cdot[(35,-7)+(7,1)]$

$= (3,4)\cdot(42,-6)$

$= 3\times42 + 4\times(-6)$

$= 102$ #

(b) $|(\vec{u}\cdot\vec{w})\vec{w}|$

$= |[(3,4)\cdot(7,1)](7,1)|$

$= |(21+4)(7,1)|$

$= |(175, 25)|$

$= 176.78$ #

(c) $|\vec{u}|(\vec{v}\cdot\vec{w})$

$= |(3,4)|\,[(5,-1)\cdot(7,1)]$

$= 5(35-1)$

$= 170$ #

(d) $(|\vec{u}|\vec{v})\cdot\vec{w}$

$= [|(3,4)|(5,-1)]\cdot(7,1)$

$= [5(5,-1)]\cdot(7,1)$

$= (25,-5)\cdot(7,1)$

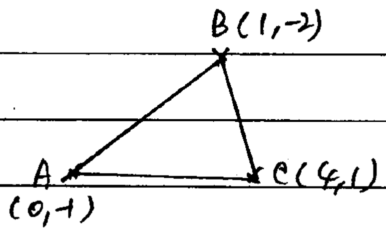
$= 25\times 7 - 5\times 1$

$= 170$ #

5. $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (1,-2) - (0,-1)$

$= (1,-1)$



$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= (4,1) - (0,-1)$

$= (4,2)$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= (4,1) - (1,-2)$

$= (3,3)$

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Use the formula $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$

Then $\cos A = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \frac{(1,-1)\cdot(4,2)}{\sqrt{2}\times\sqrt{20}} = \frac{4-2}{\sqrt{40}} = \frac{1}{\sqrt{10}}$

$\cos B = \frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|} = \frac{(-1,1)\cdot(3,3)}{\sqrt{2}\times\sqrt{18}} = \frac{-3+3}{\sqrt{36}} = \phi$

$\cos C = \frac{\overrightarrow{AC}\cdot\overrightarrow{BC}}{|\overrightarrow{AC}||\overrightarrow{BC}|} = \frac{(4,2)\cdot(3,3)}{\sqrt{20}\cdot\sqrt{18}} = \frac{12+6}{\sqrt{360}} = \frac{3}{\sqrt{10}}$

6. Given $\vec{u} = (2,K)$, $\vec{v} = (3,5)$

(a) If $\vec{u}$ is // to $\vec{v}$, then the angle between them is zero.

$\vec{u}\cdot\vec{v} = |\vec{u}|\times|\vec{v}|$

$(2,K)\cdot(3,5) = \sqrt{2^2+K^2}\cdot\sqrt{3^2+5^2}$

$6+5K = \sqrt{K^2+4}\cdot\sqrt{34}$

$34(K^2+4) = 36+60K+25K^2$

$34K^2+136 = 25K^2+60K+36$

$9K^2-60K+100=0.$

$K = \frac{60\pm\sqrt{60^2-4\times9\times100}}{18}$

$\therefore\ K = \frac{10}{3}$

(b) If $\vec{u}$ is $\perp$ to $\vec{v}$, then the angle between them is 90°.

$\vec{u}\cdot\vec{v} = \phi$

$(2,K)\cdot(3,5) = \phi$

$6+5K = \phi \quad \therefore\ K = \frac{-6}{5}$

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(c) $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\alpha$

$6+5k = \sqrt{k^2+4} \times \sqrt{34}\cos\frac{\pi}{3}$

$\therefore \quad \frac{1}{4}\times 34(k^2+4) = 36+60k+25k^2$

$34k^2+136 = 100k^2+240k+144$

$66k^2+240k+8=0$

$33k^2+120k+4=0$

$\therefore \quad k = \dfrac{-60\pm 34\sqrt{3}}{33}$

7. Show that $\vec{u}\cdot\vec{v} = \frac{1}{4}|\vec{u}+\vec{v}|^2 - \frac{1}{4}|\vec{u}-\vec{v}|^2$

$R.H.S. = \frac{1}{4}|\vec{u}+\vec{v}|^2 - \frac{1}{4}|\vec{u}-\vec{v}|^2$

$= \frac{1}{4}[(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})] - \frac{1}{4}[(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})]$

$= \frac{1}{4}(\cancel{\vec{u}\cdot\vec{u}}+\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{u}+\cancel{\vec{v}\cdot\vec{v}}) - \frac{1}{4}(\cancel{\vec{u}\cdot\vec{u}}-\vec{u}\cdot\vec{v}-\vec{v}\cdot\vec{u}+\cancel{\vec{v}\cdot\vec{v}})$

$= \frac{1}{4}(2\vec{u}\cdot\vec{v}+2\vec{v}\cdot\vec{u})$

$= \frac{1}{4}(4\times\vec{u}\cdot\vec{v})$

$= \vec{u}\cdot\vec{v}$ #

8. Unit vector in the direction of $i-j+2k$ is

$$\frac{i-j+2k}{\sqrt{1^2+1^2+2^2}} = \frac{i-j+2k}{\sqrt{6}}.$$

$\therefore$ The projection of the vector $7i+2j-k$ in the direction $i-j+2k$ is $(7i+2j-k)\cdot\left(\dfrac{i-j+2k}{\sqrt{6}}\right)$

$= \frac{7}{\sqrt{6}} - \frac{2}{\sqrt{6}} - \frac{2}{\sqrt{6}}.$

$= \frac{3}{\sqrt{6}}$

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9. Given $\vec{u}=(3,2,-1)$, $\vec{v}=(0,2,-3)$, $\vec{w}=(2,6,7)$

(a) $\vec{u}\times(\vec{v}\times\vec{w})$

$$\vec{v}\times\vec{w} = \begin{vmatrix} i & j & k \\ 0 & 2 & -3 \\ 2 & 6 & 7 \end{vmatrix} = 32i-6j-4k \quad \text{or } (32,-6,-4)$$

Then

$$\vec{u}\times(\vec{v}\times\vec{w}) = \begin{vmatrix} i & j & k \\ 3 & 2 & -1 \\ 32 & -6 & -4 \end{vmatrix} = -14i-20j-82k.$$

(b) $(\vec{u}\times\vec{v})\times\vec{w}$

$$\vec{u}\times\vec{v} = \begin{vmatrix} i & j & k \\ 3 & 2 & -1 \\ 0 & 2 & -3 \end{vmatrix} = -4i+9j+6k.$$

Then

$$(\vec{u}\times\vec{v})\times\vec{w} = \begin{vmatrix} i & j & k \\ -4 & 9 & 6 \\ 2 & 6 & 7 \end{vmatrix} = 27i+40j-6k$$

$-42k$

(c) $\vec{u}\times(\vec{v}-2\vec{w})$

$$\vec{v}-2\vec{w} = (0,2,-3)-2(2,6,7)$$
$$= (-4,-10,-17)$$

Then

$$\vec{u}\times(\vec{v}-2\vec{w}) = \begin{vmatrix} i & j & k \\ 3 & 2 & -1 \\ -4 & -10 & -17 \end{vmatrix} = -44i+55j-22k.$$

10. $\vec{u}=(-6,4,2)$ $\quad \vec{v}=(3,1,5)$

Then

$$\vec{u}\times\vec{v} = \begin{vmatrix} i & j & k \\ -6 & 4 & 2 \\ 3 & 1 & 5 \end{vmatrix} = 18i+36j-18k.$$

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11. Given $\vec{u} = (1, -1, 2)$ & $\vec{v} = (0, 3, 1)$. Then the area of the parallelogram is given by $|\vec{u} \times \vec{v}|$

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 0 & 3 & 1 \end{vmatrix} = -7i - j + 3k$$

$\therefore$ the area of the parallelogram is $\sqrt{(-7)^2 + (-1)^2 + 3^2} = \sqrt{59}$

12 (a) $\vec{v} \times \vec{w}$ will produce a vector which is perpendicular to vectors $\vec{v}$ & $\vec{w}$.

Besides, the projection of $\vec{u}$ on the vector $\vec{v} \times \vec{w}$ (or actually the height of the parallelepiped) is given by

$$\vec{u} \cdot \frac{\vec{v} \times \vec{w}}{|\vec{v} \times \vec{w}|}.$$

Therefore, the volume of the parallelepiped is

$$|\vec{v} \times \vec{w}| \cdot \left(\vec{u} \cdot \frac{\vec{v} \times \vec{w}}{|\vec{v} \times \vec{w}|} \right)$$

or $\vec{u} \cdot (\vec{v} \times \vec{w})$

(b) Volume $= \vec{u} \cdot (\vec{v} \times \vec{w})$

Firstly we find

$$\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 0 & 4 & -2 \\ 2 & 2 & -4 \end{vmatrix} = -12i - 4j - 8k.$$

$$\therefore \vec{u} \cdot (\vec{v} \times \vec{w}) = (2i - 6j + 2k) \cdot (-12i - 4j - 8k)$$

$$= -24 + 24 - 16$$

$$= -16$$

$\therefore$ the volume is 16.

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13. Let $\vec{r} = (x, y, z)$ be the position vector of any arbitrary point on the line, we have

$$\vec{r} = \vec{u} + k(\vec{v} - \vec{u}) \quad \text{where } k \text{ is a scalar}$$

$$= (5, 3, -2) + k(-2, 4, 1)$$

The cartesian form is

$$(x, y, z) = (5, 3, -2) + k(-2, 4, 1)$$

$$x = 5 - 2k \; ; \; y = 3 + 4k \; ; \; z = -2 + k$$

or $$\frac{x-5}{-2} = \frac{y-3}{4} = \frac{z+2}{1} = k.$$

14. Given $A = (1, 0, 1)$, $B = (-2, 5, 0)$ $C = (3, 1, 1)$

(a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (-3, 5, -1)$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (2, 1, 0)$

(b) $$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ -3 & 5 & -1 \\ 2 & 1 & 0 \end{vmatrix} = i - 2j - 13k$$

(c) Using the formula $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

$\therefore$ The vector equation of the plane is

$$\vec{r} \cdot (i - 2j - 13k) = (i + k) \cdot (i - 2j - 13k)$$
$$= -12$$

Then the cartesian form is

$$(xi + yj + zk) \cdot (i - 2j - 13k) = -12$$
$$x - 2y - 13z = -12$$