1 EDUC-250 Mathematical Analysis I

1.1 Exercise I

1. (**Convergence**). Given a sequence $(x_n)_{n\geq 1}$ of real numbers, prove that the followings are equivalent:

   (a) a is the limit of the sequence $(x_n)_{n\geq 1}$

   (b) ($\varepsilon - N$ **definition**). For any given $\varepsilon > 0$ there exists a natural number N such that $|x_n - a| < \varepsilon$ for all $n \geq N$.

   (c) (**Open Neighborhood definition**). For any $\varepsilon > 0$, there are only finitely many terms in the sequence (x_n) lying outside the neighborhood $(a-\varepsilon, a+\varepsilon)$ of the limit a.

   (d) (**Subsequence form**). Every subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of (x_n) converges to a.

   Proof. (a) $\Leftrightarrow$ (b). (b) is the definition of the concept of (a).
   (b) $\Leftrightarrow$ (c). As $|x_n - a| < \varepsilon \iff -\varepsilon < x_n - a < \varepsilon \iff a - \varepsilon < x_n < a + \varepsilon \iff x_n \in (a-\varepsilon, a+\varepsilon)$.
   If (b) holds, then for any given $\varepsilon > 0$, then
   $\{N, N+1, N+2, \cdots\} \subset \{ k \mid x_k \in (a-\varepsilon, a+\varepsilon) \}$. Hence we have,
   $\{ k \mid x_k \notin (a-\varepsilon, a+\varepsilon) \} \subset \mathbb{N}\setminus\{N, N+1, N+2, \cdots\} = \{1, 2, \cdots, N-1\}$.
   In particular, (c) holds.
   Suppose that (c) holds, then for any given $\varepsilon > 0$ the following index set $\{ k \mid x_k \notin (a-\varepsilon, a+\varepsilon) \}$ is finite, so it is bounded, and hence the maximal element N exists. Then for any natural number $n \geq N+1$, we have $x_n \notin (a-\varepsilon, a+\varepsilon)$ does not hold, this means that $x_n \in (a-\varepsilon, a+\varepsilon)$ for all $n \geq N+1$. So (b) holds.
   (b) $\Leftrightarrow$ (d). Suppose (d) holds, then the original sequence is its own subsequence, then it follows from (d) that it is a convergent sequence with limit a, and hence (b) holds.
   Suppose (b) holds, let (x_{n_k}) be a subsequence of (x_n) then we have $n_k \geq k$ for all $k \in \mathbb{N}$. It remains to show that any subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of (x_n) converges to the limit a. Since (b) holds, for any given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any natural number $n \geq N$, we have $|x_n - a| < \varepsilon$. It follows from $n_k \geq k$ that $|x_{n_k} - a| < \varepsilon$ for all $k \geq N$. So we have $\lim\limits_{k\to\infty} x_{n_k} = a$.

2. (**Divergence**). Let (x_n) be a sequence. Prove that the followings are equivalent:

   (a) The sequence (x_n) does not converge to $x \in \mathbb{R}$.

   (b) There exists an $\varepsilon_0 > 0$ such that for any $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $n_k \geq k$ and $|x_{n_k} - x| \geq \varepsilon_0 > 0$.

   (c) There exists an $\varepsilon_0 > 0$ and a subsequence (x_{n_k}) such that $|x - x_{n_k}| \geq \varepsilon_0$ for all $k \in \mathbb{N}$.

   (d) There exists a subsequence (x_{n_k}) of (x_n) which does not converges to x.

   Proof. (a) $\Leftrightarrow$ (b): It follows from the definition that (x_n) converges to the limit x
   $\iff \forall\varepsilon > 0 \ (\ \exists N \in \mathbb{N} \ (\ \forall n \geq N \ [\ |x_n - x| < \varepsilon \ ] \ ) \ )$.
   So we have (x_n) *does not converge to* the limit x
   $\iff \neg[\ \forall\varepsilon > 0 \ (\ \exists N \in \mathbb{N} \ (\ \forall n \geq N \ [\ |x_n - x| < \varepsilon \ ] \ ) \ ) \ ]$
   $\iff \exists\varepsilon > 0 \ \neg[\ \exists N \in \mathbb{N} \ (\ \forall n \geq N \ [\ |x_n - x| < \varepsilon \ ] \ ) \ ]$
   $\iff \exists\varepsilon > 0 \ [\ \forall N \in \mathbb{N} \ \neg(\ \forall n \geq N \ [\ |x_n - x| < \varepsilon \ ] \ ) \ ]$
   $\iff \exists\varepsilon > 0 \ [\ \forall N \in \mathbb{N} \ (\ \exists n \geq N \ \neg[\ |x_n - x| < \varepsilon \ ] \ ) \ ]$
   $\iff \exists\varepsilon > 0 \ [\ \forall N \in \mathbb{N} \ (\ \exists n \geq N \ [\ |x_n - x| \geq \varepsilon \ ] \ ) \ ]$.
   Rewrite the last statement again as follows, there exists $\varepsilon_0 > 0$ such that for all $k \in \mathbb{N}$ there exist $n_k \in \mathbb{N}$ with $n_k \geq k$ such that $|x_{n_k} - x| \geq \varepsilon_0$.
   (b) $\Leftrightarrow$ (c): (c) is just a restatement of (b), given that a subsequence (x_{n_k}) of (x_n) must satisfy $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$.

3. Suppose that $a \neq 0$ and $b \in \mathbb{R}$, $S \subset \mathbb{R}$. Define a map $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = ax + b$. Prove that $(\ f[S]\ )^a = f[\ S^a\ ]$.

4. Let a, b, c be elements of an ordered field $\mathbb{F}$. Suppose that $b > a$ and $c > d$. Determine which of the following holds?

   (a) $ad + bc > ac + bd$.

   (b) $ad + bc > ab + cd$.

   (c) $ac > bd$.

   (d) $a^2 + b^2 + c^2 + d^2 > ab + bc + cd + ad$.

   (e) $a^3d + bc^3 > b^3c + ad^3$.

   (f) If $a^2 + b^2 > 0$ then $(a + b)^2 > 0$.

   (g) If $(a + b)^2 > 0$ then $a^2 + b^2 > 0$.

5. Let A and B be non-empty subsets of $\mathbb{R}$. Determine which of the following holds?

   (a) If A is bounded and $B \subset A$, then B is bounded.

   (b) If A is bounded and $A \subset B$, then B is bounded.

   (c) If A and B are bounded, then $A \cap B$, $A \cup B$, $B \setminus A$ and $A \Delta B = (A \cup B) \setminus (A \cap B)$ are all bounded.

   Repeat the same questions, if we replace 'bounded' by 'bounded above' and 'bounded below' respectively.

6. Let A and B be non-empty subsets of $\mathbb{R}$. Suppose that $a < b$ for all $a \in A$ and $b \in B$. Determine which of the following holds?

   (a) $A \cap B \neq \emptyset$.

   (b) A is bounded above (bounded below).

   (c) B is bounded above (bounded below).

   (d) $\sup A \leq b$ for all $b \in B$.

   (e) $a \leq \inf B$ for all $a \in A$.

   (f) $\sup A \leq \inf B$.

   (g) $\inf A \leq \sup B$.

7. Let A and B be non-empty subsets of $\mathbb{R}$. Suppose that $a < b$ for all $a \in A$ and $b \in B$.

   (a) Prove that if $A \cap B \neq \emptyset$, then $\sup B = \inf B$.

   (b) Give subsets A and B such that $A \cap B = \emptyset$ and $\sup A < \inf B$.

   (c) Give subsets A and B such that $A \cap B = \emptyset$ and $\sup A = \inf B$.

8. (a) Let a, b, c be elements of an ordered field such that $a > b$ and $c > d$. Prove that $a + c > b + d$.

   (b) Let a, b, c be elements of an ordered field such that $a > b > 0$ and $c > d > 0$. Prove that $ac > bd$.

9. (a) Let a be a non-zero element of an ordered field K. Prove that $a^2 > 0$.

   (b) Let n be a non-zero natural number and K be an ordered field with unity element 1_k, i.e. multiplicative identity in K. Prove that $n \cdot 1_K = \underbrace{(1_K + 1_K + \cdots + 1_K)}_{n \text{ times of } 1_K}$.

10. Let A, B, C be three non-empty subsets of $\mathbb{R}$.

    (a) Suppose that A is bounded and $B \subset A$. Prove that B is bounded and $\sup B \leq \sup A$.

    (b) Suppose that $A \cap C$ is non-empty and that A and C are bounded. Prove that $A \cap C$ is bounded and that $\sup(\ \inf A, \inf C\ ) \leq \inf(\ A \cap C) \leq \sup(A \cap C) \leq \inf(\sup A, \sup C)$.

    (c) Suppose that $a \leq b$ for all $a \in A$ and $b \in B$. Prove that $\sup A \leq \inf B$. Give an example that the equality does not hold.

    (d) Suppose that A and B are bounded and define $A + B = \{\ a + b \in \mathbb{R} \mid a \in A,\ b \in B\ \}$. Prove that $A + B$ is bounded and that $\inf(A + B) = \inf A + \inf B$ and $\sup(A + B) = \sup A + \sup B$.

2 Exercise II

1. (a) Let $I_n = [0, \frac{1}{n}]$ for all $n \in \mathbb{N}$. Prove that if $x > 0$, then $x \notin \bigcap_{n\in\mathbb{N}} I_n$. Determine $\bigcap_{n\in\mathbb{N}} I_n$.

   (b) Let $J_n = (0, \frac{1}{n})$ for all $n \in \mathbb{N}$. Prove that $\bigcap_{n\in\mathbb{N}} J_n = \emptyset$.

   Recall that $x \in [a, b] \iff a \le x \le b$.

   Solution

(a) If $x > 0$ then by Archimedean Order Property there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < x$. Then $x \notin [0, \frac{1}{n}] = I_n$. Hence $x \notin \bigcap_{n\in\mathbb{N}} I_n$.
If $x < 0$ then $x \notin I_n$ for all $n \in \mathbb{N}$. In particular, $x \notin \bigcap_{n\in\mathbb{N}} I_n$.
Obviously, $0 \in I_n$ for all $n \in \mathbb{N}$, so that $0 \in \bigcap_{n\in\mathbb{N}} I_n$. In this case, $\{0\} = \bigcap_{n\in\mathbb{N}} I_n$.

2. Suppose that $a, b, c, d \in \mathbb{F}$.

   (a) If $(a, b) \cap (c, d) \ne \emptyset$, prove that $c < b$ and $a < d$.

   (b) Suppose that a, b satisfies the following condition:

   If $x \in \mathbb{F}$ such that $x > a$, then $x > b$.

   Prove that $b \le a$.

   (c) Let $c < d$, and $a < b$. If $(a, b) \subset (c, d)$, and prove that $c \le a < b \le d$.

   Proof.

(a) Since $(a, b) \cap (c, d) \ne \emptyset$, there exists $x \in (a, b) \cap (c, d)$. It follows from definition that $a < x < b$ and $c < x < d$. Then $c < x < b$ and $a < x < d$.

(b) Suppose contrary that $b > a$. Set $y = \frac{b+a}{2}$, then $y = \frac{b+a}{2} > \frac{a+a}{2} = a$. If follows from $y > a$ that $y > b$, but the latter result violates $b = \frac{b+b}{2} > \frac{b+a}{2} = y$.

(c) Suppose contrary, then if follows from $a < b$ then that $c > a$ or $b > d$.
If $c > a$, then choose $x = a + \frac{c-a}{2}$, we have $x = \frac{c+a}{2} > \frac{a+a}{2} = a$, and $x = \frac{c+a}{2} < \frac{c+c}{2} = c < b$, so $a < x < c$. Now compare the values x and b as follows.

   (i) If $x < b$ then $a < x < b$ hence $x \in (a, b)$. It follows from $(a, b) \subset (c, d)$ that $x \in (c, d)$. In particular, $c < x$, which violates $x < c$.

   (ii) If $x \ge b$, then $a < b \le x < c$. It follows that $(a, b) \cap (c, d) = \emptyset$, which violates the assumption $(a, b) \subset (c, d)$.

   We leave the remaining case $b > d$ to the reader as an exercise.

3. (i) Prove that if $A \subset B$ and B is bounded, then A is bounded.

   (ii) Prove that $A \cup B$ and $A \cap B$ are bounded; and

   (iii) Prove that $\sup(A \cup B) = \sup\{\sup A, \sup B\}$, and $\inf(A \cup B) = \inf\{\inf A, \inf B\}$.

   Proof.

(i) Since B is bounded, there exist $M > 0$ such that $|x| \le M$ for all $x \in B$. So it follows from $A \subset B$ that if $x \in A \Rightarrow x \in B \Rightarrow |x| \le M$. In particular, A is bounded.

(ii) Let $M = \sup\{\sup A, \sup B\}$. Then $\sup A \le M$ and $\sup B \le M$.

   $x \in A \cup B \iff (x \in A) \vee (x \in B) \Rightarrow (x \le \sup A) \vee (x \le \sup B) \Rightarrow (x \le M) \vee (x \le M) \iff x \le M$. So $A \cup B$ is bounded above. Similarly, one can prove that $x \in A \cup B \Rightarrow x \ge \inf\{\inf A, \inf B\}$, and hence $A \cup B$ is bounded below. Finally, $A \cap B$ is a subset of a bounded set $A \cup B$, so if follows from (i) that $A \cap B$ is bounded too.

(iii) We only prove that $\sup(A \cup B) = \sup\{\sup A, \sup B\}$. It follows from the proof of (ii) that $\sup\{\sup A, \sup B\}$ is an upper bound of $A \cup B$. For any $\varepsilon > 0$, consider the following two cases: (1) If $\sup(A \cup B) = \sup A$ then there exists $x \in A$ such that $x > \sup A - \varepsilon$, in particular, $x \in A \cup B$ such that $x > \sup A - \varepsilon$, hence $\sup(A \cup B) = \sup A = \sup\{\sup A, \sup B\}$. (2) If

$\sup(A \cup B) = \sup B$ then there exists $y \in B$ such that $y > \sup B - \varepsilon$, in particular, $y \in A \cup B$ such that $y > \sup B - \varepsilon$, hence $\sup(A \cup B) = \sup B = \sup\{\sup A, \sup B\}$.

The remaining case for infimum is left as an exercise.

4. Determine the supremum and infimum of the following sets:
$A = \{ 3(1 - \frac{1}{n}) + 2(-1)^n \mid n \in \mathbb{N}\}$, $B = \left\{ \frac{1}{n} + \frac{1}{m} \;\middle|\; n, m \in \mathbb{N} \right\}$ and $C = \left\{ \frac{m}{nm+1} \;\middle|\; n, m \in \mathbb{N} \right\}$.
Proof.

A. Let $E = \left\{ 1 - \frac{3}{2n-1} \;\middle|\; n \in \mathbb{N} \right\}$ and $O = \left\{ 5 - \frac{3}{2n} \;\middle|\; n \in \mathbb{N} \right\}$. Since $\left(1 - \frac{3}{2n-1}\right)$ and $(5 + \frac{3}{2n})$ are monotone increasing sequence. Thus $\sup E = 1$ and $\sup O = 5$. Thus $\inf E = -2$ and $\inf O = \frac{7}{2}$. We have $A = E \cup O$. In this case, $\sup A = \max\{ \sup E, \sup O \} = 5$. Similarly $\inf A = \min\{ \inf E, \inf O \} = -2$.

B. Since $B = \left\{ \frac{1}{n} + \frac{1}{m} \;\middle|\; n, m \in \mathbb{N} \right\} = P + P$, where $P = \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N} \right\}$ with $\sup P = 1$ and $\inf P = 0$. It follows that $\sup B = \sup(P + P) = \sup P + \sup P = 1 + 1 = 2$. Similarly, $\inf B = \inf(P + P) = \inf P + \inf P = 0 + 0 = 0$.

C. Since $\frac{m}{nm+1} \le \frac{m}{nm} = \frac{1}{n} \le 1$, for all $n, m \in \mathbb{N}$. Then 1 is an upper bound of C. For any $\varepsilon > 0$ then $\frac{1}{\varepsilon} - 1 \in \mathbb{R}$, it follows from Archimedean Order Property that there exists $m_0 \in \mathbb{N}$ such that $m_0 > \frac{1}{\varepsilon} - 1$. And $m_0 > \frac{1}{\varepsilon} - 1 \iff \frac{m_0}{m_0(1)+1} > 1 - \varepsilon$. Thus $\sup C = 1$.
Since $0 \le \frac{m}{nm+1}$ so 0 is a lower bound of C. For any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} \le \varepsilon$, it follows that $\frac{1}{n_0(1)+1} \le \frac{1}{n_0} \le 0 + \varepsilon$. So $\inf C = 0$.

5. If (x_n) is a monotone increasing sequence with limit a, prove that $\sup\{ x_n \in \mathbb{R} \mid n \in \mathbb{N}\} = a$.
Hint: prove that a is an upper bound of the set $\{ x_n \in \mathbb{R} \mid n \in \mathbb{N}\}$.
Proof. (i) We first show that a is an upper bound of $(x_n)_{n\in\mathbb{N}}$. Suppose contrary that a is *not* an upper bound of (x_n), then there exists $N \in \mathbb{N}$ such that $x_N > a$. In particular, for all $n \ge N$ we have $x_n \ge x_N > a$. In particular, $|x_n - a| = x_n - a$. Let $\varepsilon = x_N - a > 0$, it follows from the $\lim_{n\to\infty} x_n = a$ that there exists $M \in \mathbb{N}$ such that for all $n \ge M$ we have $|x_n - a| < \varepsilon$. Choose $m = 1 + \max\{N, M\}$ then $m > M$ and $m > N$ so $x_m - a = |x_m - a| < \varepsilon = x_N - a$. In particular, $x_m < x_M$ where $m > M$. This violates the increasing property of (x_n).
(ii) We now show that s is the least upper bound of the set $\{ x_n \in \mathbb{R} \mid n \in \mathbb{N}\}$. For any $\varepsilon > 0$, it follows from that there exists $N \in \mathbb{N}$ such that $a - x_N = |x_N - a| < \varepsilon$. So we have $x_N > a - \varepsilon$, and hence $\sup\{ x_n \in \mathbb{R} \mid n \in \mathbb{N}\} = a$.

6. If S be a non-empty bounded subset of a complete ordered field $\mathbb{F}$, and $a \in S$. If a is an upper bound of S, prove that $a = \sup S$.
Proof. It follows from the Supremum Principle that $\sup S$ exists in $\mathbb{F}$. Suppose that $a \neq S$, ie. a is not the least upper bound of S. In particular, $\sup S < a$. Since $\sup S$ is an upper bound of S we have $a \le \sup S$, which contradicts $\sup S < a$. So the result follows.

7. Determine the supremum and infimum of the set
$S = \left\{ \frac{n+1}{n^2+32} \;\middle|\; n \in \mathbb{N},\ n \neq 0 \right\}$.

Solution. Let $f(n) = \frac{n+1}{n^2+32}$, then

$$f(n) - f(n+1) = \frac{n+1}{n^2+32} - \frac{n+2}{(n+1)^2+32} = \frac{-(n^2+3n-31)}{(32+n^2)(32+(n+1)^2)}.$$

Note that the denominator $(32 + n^2)(32 + (n+1)^2)$ is positive.

(a) If $1 \le n \le 4$, the numerator $n^2 + 3n - 31 < 0$, and hence $f(n+1) > f(n)$ and so $f(5) > f(4) > f(3) > f(2) > f(1)$.

(b) If $n \geq 5$, then the numerator $n^2+3n-31>0$, and hence $f(n) > f(n+1)$ and so $f(5) > f(6) > f(7) > \cdots$. Thus $\sup S = f(5) = 2/19$.

(c) It follows from $n \geq 1$ that $n+1 \geq 1 > 0$, and $n^2+32 > 32 > 0$, so $f(n) > 0$ and that $f(n) = \dfrac{1+n}{32+n^2} \leq \dfrac{1+n}{n^2} \leq \dfrac{2n}{n^2} = \dfrac{2}{n}$. we see that 0 is the lower bound of S, and by means of Archimedean Ordering property, for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \frac{\varepsilon}{2}$. In particular, we have $f(n) < \frac{2}{n} < \varepsilon = 0 + \varepsilon$. So $\inf S = 0$.

8. Determine the supremum and infinmum of the following subsets of $\mathbb{R}$: (i) $\left\{ 1 - \dfrac{1}{n} \mid n \in \mathbb{N},\ n \neq 0 \right\}$; (ii) $\left\{ 1 + \dfrac{(-1)^n}{n} \mid n \in \mathbb{N},\ n \neq 0 \right\}$, and (iii) $\left\{ \dfrac{2n}{2n-5} \mid n \in \mathbb{N} \right\}$.

9. Let x be a strictly positive real number. Prove that there exists a strictly positive integer n such that $\dfrac{1}{n} < x < n$.

Solution. For any $x > 0$, then $0 < 1/x$. Let $M = \max\{\ x,\ \dfrac{1}{x}\ \}$, choose $n \in \mathbb{N}$ such that $M < n$. In particular, $\dfrac{1}{x} \leq M < n$ and $x \leq M < n$. So $\dfrac{1}{n} < x$. Then $\dfrac{1}{n} < x < n$.

10. Prove that the following numbers are irrational: $\sqrt{3}$, $\sqrt[3]{2}$, $\sqrt{2}+\sqrt{3}$. Is the sum of two irrational numbers still irrational?

11. Show that the set of irrational numbers is dense in $\mathbb{R}$, i.e. for any real numbers $a < b$, there exists an irrational number x such that $a < x < b$.

12. Let a, b be real numbers such that $a \leq b + \dfrac{1}{n}$ for all non-zero natural number n. Show that $a \leq b$.

Solution Suppose contrary that $a > b$. By Archimedean Order Property there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < a - b$. By assumption, we have $a \leq b + \dfrac{1}{n} < b + (a-b) = a$, which is impossible.

13. Let a be a real number, show that $\sup\{\ x \in \mathbb{R} \mid x < a\ \} = a$.

Solution. Let $S = \{\ x \in \mathbb{R} \mid x < a\ \}$, then $x < a$ for all $x \in S$. So S is bounded above. We know that $-|a| - 1 \in S$ so that S is non-empty. By supremum principle, we know $\sup S$ exists in $\mathbb{R}$, and hence $\sup S \leq a$. It remains to show $\sup S = a$. For any $\varepsilon > 0$, set $x = a - \varepsilon$, $x < a$ hence $x \in S$, and that $x < a + \varepsilon$. So $a = \sup S$.

14. (a) Prove that $A^a \cup B^a = (A \cup B)^a$.

(b) Prove that $A^a \cap B^a \subset (A \cap B)^a$. Does the equality holds?

(c) Let A_n be a family of subsets of $\mathbb{R}$. Does $\bigcup_{n=1}^{\infty} (A_n)^a = \left(\bigcup_{n=1}^{\infty} A_n\right)^a$ holds?

Hint: For $(A \cup B)^a \subset A^a \cup B^a$, one can consider $x \notin A^a \cup B^a$.

Proof.

(a) If suffices to prove that if $A \subset B$ then $A^a \subset B^a$.
Let $x \in A^a$, Then for any $\varepsilon > 0$ the set $(x-\varepsilon, x+\varepsilon) \cap A \setminus \{x\} \neq \emptyset$. It follows from $A \subset B$ that $(x-\varepsilon, x+\varepsilon) \cap A \setminus \{x\} \subset (x-\varepsilon, x+\varepsilon) \cap B \setminus \{x\}$, hence the latter is non-empty too, so $x \in B^a$.
Since $A \subset A \cup B$ so we have $A^a \cup B^a \subset (A \cup B)^a$.
Let $x \notin (A^a \cup B^a)$, then $x \notin A^a$ and $x \notin B^a$. So $\exists \delta_i > 0$ $(i = 1, 2)$ such that $(x - \delta_1, x + \delta_1) \cap A \setminus \{x\} = \emptyset$ and $(x - \delta_2, x + \delta_2) \cap B \setminus \{x\} = \emptyset$. Set $\delta = \min\{\delta_1, \delta_2\} > 0$ then it follows from the distributive law that $(x-\delta, x+\delta) \cap (A \cup B) \setminus \{x\} = (\ (x-\delta, x+\delta) \cap (A \cap B) \setminus \{x\}\ ) \cup (\ (x-\delta, x+\delta) \cap (A \cap B) \setminus \{x\}\ ) = \emptyset \cup \emptyset = \emptyset$. Hence $x \notin (A \cup B)^a$.
It follows that $(A \cup B)^a = A^a \cup B^a$.

(b) $(A \cap B)^a \subset A^a \cap B^a$ follows easily from $A \cap B$ is a subset of A and subset of B. The equality does not hold in general. Take $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ and $B = \{-\frac{1}{n} \mid n \in \mathbb{N}\ \}$. It is easy to check that $A^a = B^a = \{0\}$. But $A \cap B = \emptyset$. so $A^a \cap B^a \neq (A \cap B)^a$.

(c) Equality does not hold. For example $A_n = \{\ \frac{1}{n}\ \}$, for all $n \in \mathbb{N}$. Each of them is a finite set so $(A_n)^a = \emptyset$. Then $\bigcup_{n=1}^{\infty} (A_n)^a = \emptyset$ but $\left(\bigcup_{n=1}^{\infty} A_n\right)^a = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}^a = \{0\}$.

3 Exercise III

1. Demonstrate, by means of examples, that the following "definitions" of convergence of a sequence (u_n) are incorrect:

   (a) $\exists N \in \mathbb{N}$ such that $\forall \varepsilon > 0$, we have $|u_n - l| < \varepsilon$ for all $n \geq N$.

   (b) $\forall N \in \mathbb{N}$ such that $\exists \varepsilon > 0$, we have $|u_n - l| < \varepsilon$ for all $n \geq N$.

   (c) $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n \geq N$, we have $|u_n - l| < \varepsilon$.

   (d) $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n \geq N$, we have $|u_n - l| \leq \varepsilon$.

   (e) $\forall \varepsilon \geq 0$, $\exists N \in \mathbb{N}$ such that for all $n \geq N$, we have $|u_n - l| < \varepsilon$.

   (f) $\forall \varepsilon \geq 0$, $\exists N \in \mathbb{N}$ such that for all $n \geq N$, we have $|u_n - l| \leq \varepsilon$.

   (g) $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n \leq N$, we have $|u_n - l| < \varepsilon$.

   (h) $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n \leq N$, we have $|u_n - l| \leq \varepsilon$.

   (i) $\forall \varepsilon \geq 0$, $\exists N \in \mathbb{N}$ such that for all $n \leq N$, we have $|u_n - l| < \varepsilon$.

   (j) $\forall \varepsilon \geq 0$, $\exists N \in \mathbb{N}$ such that for all $n \leq N$, we have $|u_n - l| \leq \varepsilon$.

2. Determine which of the following statement is true or false:

   (a) Monotone decreasing sequence is convergent.

   (b) Every sequence has a monotone subsequence.

   (c) A bounded sequence (x_n) is a bounded infinite subset of $\mathbb{R}$.

   (d) A bounded subset of $\mathbb{R}$ has at least an accumulation point.

   (e) If S has non-empty S^a, then S is infinite.

   (f) If S is infinite, then $S^a \neq \emptyset$.

   (g) If S is non-empty and bounded, then $S^a \neq \emptyset$.

   (h) If S is bounded, then S^a is bounded.

   (i) If S^a is bounded, then S is bounded.

   (j) If the supremum of S exists, then S is bounded.

   (k) If a sequence (x_n) is convergent, then $\{ x_n \mid n \in \mathbb{N} \}^a$ is non-empty.

4 Exercise IV

1. Let (x_n) be a sequence, define another sequence $y_n = a$ and $y_{n+1} = x_n$, i.e. (y_n) is obtained by inserting one extra term a to the sequence (x_n) in its first term. Prove that (y_n) is convergent if and only if (x_n) convergent. In this case, their limits are the same.

2. Let (x_n) and (y_n) be two convergent sequences with the same limits a. Define another sequence (z_n) as follows: $z_{2n} = x_n$ and $z_{2n-1} = y_n$, i.e. $(z_n) = (y_1, x_1, y_2, x_2, \cdots, y_n, x_n, \cdots)$. Prove that (z_n) is a convergent sequence with limit a.

3. Let (x_n) and (y_n) be two convergent sequences with limits a and b respectively. Suppose that $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ be strictly increasing functions such that $f[\mathbb{N}]$ and $g[\mathbb{N}]$ form a partition of $\mathbb{N}$, i.e. $f[\mathbb{N}] \cap g[\mathbb{N}] = \emptyset$ and $f[\mathbb{N}] \cup g[\mathbb{N}] = \mathbb{N}$. Define $z_n = x_{f^{-1}(n)}$ if $n \in f[\mathbb{N}]$; otherwise $z_n = y_{g^{-1}(n)}$. Prove that (z_n) is convergent if and only if $a = b$.

4. Let (x_n) be a sequence such that all its subsequences $(x_{n_k})_{k \in \mathbb{N}}$, except the original one, are convergent. Prove that the limits of all subsequences are the same.

   Proof. Consider the subsequence $(x_{n+1})_{n \in \mathbb{N}}$ by deleting the first term from the original sequence. It follows from the assumption that this subsequence $(x_{n+1})_{n \in \mathbb{N}}$ is convergent and let $a = \lim_{n \to \infty} x_{n+1}$. One can insert back the x_1 to $(x_{n+1})_{n \in \mathbb{N}}$ as the first term, so that we obtain the original sequence $(x_n)_{n \in \mathbb{N}}$, so the original sequence converges to the same limit a. It follows from that every subsequence will converges to the same limit a.

5. Let $(x_n)_{n \in \mathbb{N}}$ be a convergent sequence with limit a and suppose that $x_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$. Prove that (i) the sequence (x_n) can take only finitely many values, i.e. $\{ x_n \in \mathbb{R} \mid n = 1, 2, \cdots, \}$ is a finite set; and (ii) $a \in \mathbb{Z}$.

Proof. Let $a = \lim_{n\to\infty} x_n$. Take $\varepsilon = 1/2 > 0$ then there exists $N \in \mathbb{N}$ such that for any $n \geq N$ we have $|x_n - a| < 1/3$. Then we have $a \in (x_n - 1/3, x_n + 1/3) = I_n$ for all $n \geq N$. This is $a \in \bigcap_{n\geq N} I_n$. Note that each interval I_n contains only one integer, namely x_n.

Suppose that there exist $n, m \in \mathbb{N}$ such that $x_n \neq x_m$, then it follows from $x_n \in \mathbb{Z}$ that $|x_n - x_m| \geq 1$. We want to prove that these two intervals $(x_n - 1/2, x_n + 1/2)$ and $(x_m - 1/2, x_m + 1/2)$ are disjoint as follows. Compare the end points of intervals $I_n = (x_n - 1/2, x_n + 1/2)$ and $I_m = (x_m - 1/3, x_m + 1/3)$ as follows: $|(x_m - \frac{1}{3}) - (x_n + \frac{1}{3})| = |x_m - x_n - 1| \geq |x_n - x_m| - 2/3 \geq 1/3 > 0$ and similarly, $|(x_m + \frac{1}{3}) - (x_n - \frac{1}{3})| \geq 1/3 > 0$. But both intervals have length 2/3, so the result follows.

As $a \in \bigcap_{n\geq N} I_n$, it follows from the discussion above that $x_n = x_N$ for any $n \geq N$. Then the subsequence $(x_n)_{n\geq N}$ is a constant sequence and hence it converges to x_N. As a subsequence of the original convergent sequence $(x_n)_{n\in\mathbb{N}}$. we know that their limits are the same, so $a = x_N \in \mathbb{Z}$, and hence $(x_n)_{n\in\mathbb{N}}$. takes on only at most N values, namely, these N values from the set $\{x_1, x_2, \cdots, x_{N+1}, x_N = a\}$.

6. Determine the limits of the following sequences (if they exist):
   (a) $u_n = \sqrt{n^2+1} - n$; (b) $v_n = \sqrt{n^2+n} - n$.
   Proof. $\lim_{n\to\infty} u_n = 0$. and $\lim_{n\to\infty} v_n = \frac{1}{2}$.

(a) As$\sqrt{n^2+1} + n > \sqrt{n^2} + n = 2n$, we have $u_n = \sqrt{n^2+1} - n = \dfrac{n^2+1-n^2}{\sqrt{n^2+1}+n} = \dfrac{1}{\sqrt{n^2+1}+n} < \dfrac{1}{2n}$. It follows from Squeeze theorem and that $\lim_{n\to\infty} \frac{1}{2n} = 0$ that $0 \leq \lim_{n\to\infty} u_n \leq \lim_{n\to\infty} 1/(2n) = 0$.

(b) As $\lim_{n\to\infty} \left(\sqrt{1+\frac{1}{n}} + 1\right) = \sqrt{1 + \lim_{n\to\infty} \frac{1}{n}} + 1 = 1 + \sqrt{1} = 2 > 0$, so we have

$$v_n = \sqrt{n^2+n} - n = \frac{\sqrt{n^2+n}^2 - n^2}{\sqrt{n^2+n}+n} = \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+\frac{1}{n}}+1} \to \frac{1}{2}$$

as $n \to \infty$. (Give reasons.) This method explain how you can apply the theorem of limits to find the limit of a sequence.

Below you can verify the limit 1/2 from the simple estimation, for any given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\frac{1}{2n} < \varepsilon$ so we have $|v_n - \dfrac{1}{2}| = \left|\dfrac{n}{\sqrt{n^2+n}+n} - \dfrac{1}{2}\right| = \left|\dfrac{n - \sqrt{n^2+n}}{2(n+\sqrt{n^2+n})}\right| = \left|\dfrac{(n^2+n)-n^2}{2(n+\sqrt{n^2+n})^2}\right| < \dfrac{n}{2n^2} = \dfrac{1}{2n} < \varepsilon$.

7. Prove that the sequence $\{ \sqrt{n^2+n} - n \mid n \in \mathbb{N} \}$ is bounded from below by $\frac{1}{3}$.
   Proof I. As $n \in \mathbb{N}$, we have $5n \geq 5 > 4$, then $9n > 4n + 4$ and hence $3\sqrt{n} > 2\sqrt{n+1}$ and finally $\frac{\sqrt{n}}{2\sqrt{n+1}} > \frac{1}{3}$. So we have $v_n = \sqrt{n^2+n} - n = \dfrac{\sqrt{n^2+n}^2 - n^2}{\sqrt{n^2+n}+n} = \dfrac{n}{\sqrt{n^2+n}+n} = \dfrac{\sqrt{n}}{\sqrt{n+1}+1} > \dfrac{\sqrt{n}}{2\sqrt{n+1}} > \dfrac{1}{3}$.

8. Determine the limits of the following sequences (if they exist):
   (a) $a_n = \frac{n}{n+1}$; (b) $b_n = \frac{(-1)^n}{n}$; (c) $c_n = \frac{3n^2+5}{2n^2-4}$; (d) $d_n = \frac{2n^2+3}{n-1}$;
   (e) $e_n = \frac{n+\sqrt{n}+1}{n-3}$; (f) $f_n = \frac{n+(-1)^n\sqrt{n}+1}{n-3}$; (g) $g_n = \frac{(-1)^n n^2+n-3}{2n^2+n-4}$.
   Proof. In the following, we use the theorem about limit to find out the limit first, then we verify the limit by means of ε–N method.
   $c_n = \dfrac{3n^2+5}{2n^2-4} = \dfrac{3+\frac{5}{n^2}}{2-\frac{4}{n^2}} \to \dfrac{3}{2}$, as $n \to \infty$. (Give reasons.)

$$|c_n - \frac{3}{2}| = \left|\frac{3n^2+5}{2n^2-4} - \frac{3}{2}\right| = \left|\frac{(6n^2+10)-(2n^2-12)}{2(2n^2-4)}\right| = \frac{11}{2n^2-4}.$$

   Working backward $2n^2 - 4 > 2(n-1)^2 \iff 2n^2 - 4 > 2n^2 - 4n + 2 \iff 4n > 2$, which is obvious as $n \geq 1$.

   For any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $N - 1 > \frac{11}{2\varepsilon}$ then for any natural number $n > N$ we have $\left|c_n - \dfrac{3}{2}\right| = \dfrac{11}{2n^2-4} < \dfrac{11}{2(n-1)^2} < \dfrac{11}{2(n-1)} \leq \dfrac{11}{2(N-1)} < \varepsilon$.

9. Let (u_n) be a sequence of positive numbers converging to 0. Show that $(\sqrt{u_n})$ also converges to 0.
   Proof. For any $\varepsilon > 0$, then $\varepsilon^2 > 0$, it follows from $\lim_{n\to\infty} u_n = 0$ that there

exists $N\in\mathbb{N}$ such that $|u_n - 0| < \varepsilon^2$ for all $n \geq N$. In particular, for all $n \geq N$ we have $|\sqrt{u_n} - \sqrt{0}| = \sqrt{u_n} < \sqrt{\varepsilon^2} = \varepsilon$. So we have $\lim\limits_{n\to\infty} \sqrt{u_n} = 0$.

10. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence converging to $a \in \overline{\mathbb{R}} = \mathbb{R}\cup\{+\infty\}$. Define sequence $(b_n)_{n\in\mathbb{N}}$ as follows: $b_n = \dfrac{u_1 + \cdots + u_n}{n}$ for all $n\in\mathbb{N}$. Prove that $\lim\limits_{n\to\infty} b_n = a$.

Proof. One divide into 2 cases as follows:

(i) If $\lim\limits_{n\to\infty} u_n = \infty$, then for any given $M > 0$ there exists $N\in\mathbb{N}$ such that for any $n \geq N$ we have $a_n \geq M+1$. Let $C = u_1 + u_2 + \cdots + u_N$. As $1/2 > 0$ there exists $K\in\mathbb{N}$ so that $\frac{|C-N(M+1)|}{K} < \frac{1}{2}$.

So for any $n \geq K$ we have $b_n = \dfrac{u_1+\cdots+u_n}{n} \geq \dfrac{C+(n-N)(M+1)}{n} = M + \left(1 + \dfrac{C-N(M+1)}{n}\right) = M + \left|1 - \dfrac{|C-N(M+1)|}{n}\right| > M$. It follows that $\lim\limits_{n\to\infty} b_n = \infty$.

(ii) If $\lim\limits_{n\to\infty} u_n = a\in\mathbb{R}$, then for any $\varepsilon > 0$ there exists $N\in\mathbb{N}$ such that for any $n \geq N$ we have $|u_n - a| < \varepsilon/2$, i.e. $a - \varepsilon/2 < u_n < a + \varepsilon/2$. Let $C = u_1 + u_2 + \cdots + u_N$. As $|C - N(a+\varepsilon/2)| \geq 0$ there exists a natural number $K \geq N$ such that $K \geq \dfrac{2|C-N(a+\varepsilon/2)|}{\varepsilon}$. Then $b_n = \dfrac{u_1+\cdots+u_n}{n} = \dfrac{C+(u_{N+1}+\cdots+u_n)}{n} < \dfrac{C+(n-N)(a+\varepsilon/2)}{n} = (a+\dfrac{\varepsilon}{2}) + \dfrac{C-N(a+\varepsilon/2)}{n} \leq (a+\dfrac{\varepsilon}{2}) + \dfrac{|C-N(a+\varepsilon/2)|}{n} \leq (a+\dfrac{\varepsilon}{2}) + \dfrac{|C-N(a+\varepsilon/2)|}{K} \leq a + \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} = a+\varepsilon$.

Similarly, one can choose $L > K$ such that $\dfrac{2|C-N(a-\varepsilon/2)|}{\varepsilon} < L$. Then for any $n \geq L$ we have

$$b_n = \frac{C+(u_{N+1}+\cdots+u_n)}{n} > \frac{C+(n-N)(a-\varepsilon/2)}{n}$$
$$= (a-\frac{\varepsilon}{2}) + \frac{C-N(a-\varepsilon/2)}{n} \geq (a-\frac{\varepsilon}{2}) - \frac{|C-N(a-\varepsilon/2)|}{n}$$
$$\geq (a-\frac{\varepsilon}{2}) - \frac{|C-N(a-\varepsilon/2)|}{L} \geq a - \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = a - \varepsilon.$$

Consequently, for any natural number $n \geq L$ we have $b_n \in (a-\varepsilon, a+\varepsilon)$, and hence the result follows.

11. Use the result of previous exercise to determine the limit $\lim\limits_{n\to\infty} \sqrt[n]{(n!)}$.

12. One define a sequence (u_n) as follows: $u_1 = 1$ and $u_{n+1} = 2u_n^2$.

   (a) Show that if the sequence (u_n) converges, then the limit is 0 or 1/2.

   (b) Does the sequence (u_n) converges?

Proof.

(a) If (u_n) converges to a then its subsequence (u_{n+1}) converges to a as well. Then it follows that (u_{n+1}^2) converges to a^2. Passing to limit, we have $a = \lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} 2u_n^2 = 2(\lim\limits_{n\to\infty} u_n)^2 = 2a^2$, i.e. $a = 2a^2$. So it follows that $a(2a-1) = 0$ then $a = 0$ or $1/2$.

(b) No, the sequence (u_n) does not converge.
One can show, by mathematical induction that $u_n \geq 2^{n-1}$, for all $n \geq 1$. When $n = 1$ we have $u_1 = 1 \geq 2^0$. Suppose that $u_n \geq 2^{n-1}$ it follows that $u_{n+1} = 2u_n^2 = 2\cdot(2^{n-1})^2 \geq 2\cdot 2^{n-1} = 2^n = 2^{(n+1)-1}$.
If follows from $u_n \geq 2^{n-1}$ that the sequence (u_n) is not bounded below, and hence it is divergent.

13. (a) What does it mean to say that a sequence converges to a limit l?

   (b) State the monotone convergence axiom.

   (c) For the following two statements, provide proofs or counterexamples:

      i. Suppose that $a_n \leq b_n$ for every n and $a_n \to a$, them $a \leq b$.

      ii. Suppose that $a_n < b_n$ for every n and $a_n \to a$, them $a < b$.

   (d) Let $a_n = \dfrac{3n^2+4}{n^2+8n+7}$. State carefully any theorems you use about limits, prove that a_n converges to 3.