EDUC 250 Mathematical Analysis   Test One

Due: 26th October, 2004. Hand in before the lecture starts at 9:00 a.m.

1. Let $A = \{a_1, a_2, \cdots, a_n\}$ and $B = \{b_1, b_2, \cdots, b_n\}$ be subsets of an ordered field $\mathbb{F}$. Define

   (i) $A + B = \{\ a_i + b_j \mid i, j = 1, 2, \cdots, n\ \}$;

   (ii) $A - B = \{\ a_i - b_j \mid i, j = 1, 2, \cdots, n\ \}$;

   (iii) $A \cdot B = \{\ a_i b_j \mid i, j = 1, 2, \cdots, n\ \}$;

   (iv) $A/B = \{\ a_i/b_j \mid i, j = 1, 2, \cdots, n,$ and $b_j \neq 0\ \}$.

   Let $M_A = \max A$, $M_B = \max B$, $m_A = \min A$ and $m_B = \min B$. Answer the following questions with correct answer, no proof is needed.

   (a) Give the formula of $supA$ and $\inf A$ in terms of M_A, m_A.

   (b) If $\sup A = \inf A$, what can you say about A?

   (c) Give the formula of the maximum element of $A - B$ in terms of the M_A, M_B, m_A and m_B.

   (d) Give two subsets A and B such that $\max(A \cdot B) \neq \max A \cdot \max B$.

   (e) Give the formula of the maximum element of $A \cdot B$ in terms of the M_A, M_B, m_A and m_B.

   (f) Give the formula of the maximum element of A/B in terms of the M_A, M_B, m_A and m_B, provided $b \neq 0$ for all $b \in B$.

   (g) (Bonus) Give the formula of the maximum element of $(A \cup \{0\}) - (A \cup \{0\})$ in terms of M_A, m_A.

2. (a) State the definitions of a *bounded* subset B, and that of an *unbounded* subset U of an ordered field $\mathbb{F}$.

   (b) Give an example of *unbounded* subset of the real number field $\mathbb{R}$. Justify your claim.

   (c) Let A, B be non-empty, bounded subsets of an ordered field $\mathbb{F}$. Prove that $A \cap B$ and $A \cup B$ are bounded subsets of $\mathbb{F}$.

   (d) Let n be a fixed natural number, A be a subset of positive elements of an ordered field. Define $A^n = \{x_1 \cdot x_2 \cdots x_n \mid x_i \in A\ (1 \leq i \leq n)\ \}$. Prove that A^n is bounded above if and only if A is bounded above.

3. (a) State the Supremum Principle.

   (b) Suppose that $\mathbb{F}$ is an ordered field satisfying supremum principle, prove that a non-empty, bounded below subset of $\mathbb{F}$ has an infimum.

   (c) (Bonus) Let $\mathbb{Q}$ be the field of rational numbers with the usual addition, multiplication and ordering. Does the supremum principle hold for $\mathbb{Q}$? Give an explanation for your answer.

4. Let A be a non-empty, bounded above subset of an ordered field $\mathbb{F}$. Suppose that $\sup A$ exists in $\mathbb{F}$ and l is an upper bound of A.

   (a) Is $|l|$ is an upper bound of A?

   (b) State a necessary and sufficient condition that $l \neq \sup A$. Prove that your answer is correct.

   (c) Prove that $l = \sup A$ if and only if for any $n \in \mathbb{N}$, the number $l - \frac{1}{n}$ is not an upper bound of S.

   (d) (Bonus) Suppose that $\sup(A \setminus \{l\}) = l$, prove that for any $\varepsilon > 0$ the set $A \cap (l - \varepsilon, l)$ is non-empty and is not finite.

5. Let $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ be the set of all natural numbers, integers, rational numbers, real numbers respectively. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \dfrac{2x}{1 + x^2}$ for all $x \in \mathbb{R}$.
   Determine $\sup f[X_i]$ and $\inf f[X_i]$ when $X_1 = \mathbb{R}$; $X_2 = \mathbb{Q}$; $X_3 = \mathbb{Z}$ and $X_4 = \mathbb{N}$.
   Hint: Recall that $f[S] = \{\ f(s) \mid s \in S\ \}$, and you may use trigonometric functions, or some famous inequalities.