EDUC 250 Mathematical Analysis Solution of Homework I

Let $\mathbb{F}$ be a field, $a, b, c, d \in \mathbb{F}$. Prove that the followings hold:

1. $-(-a) = a$.

   Proof. By definition, $-(-a) + (-a) = 0$, and $a + (-a) = 0$. By the uniqueness of solution of equation $x + (-a) = 0$, we have $-(-a) = a$.

2. $-(a - b) = b - a$.

   Proof. Let $x = -(a - b)$, by definition, we have $x + (a - b) = 0$. Hence $x+a = x+a+0 = x+a+(-b+b) = (x+(a-b)\ )+b = 0+b = b$. And $x = x + 0 = x + (a - a) = (x + a) + (-a) = b - a$. Then by uniqueness, we have $-(a - b) = b - a$.

3. $a \cdot (-b) = -(a \cdot b) = (-a) \cdot b$.

   Proof. Let $y = -b$, then $y+b = 0$. Multiplying by a, we have $a\cdot y+a\cdot b = 0$. Hence we have we have $-(a \cdot b) = ay = a \cdot (-b)$. Similarly, we have $(-a) \cdot b = b \cdot (-a) = -(b \cdot a) = -(a \cdot b)$.

   Remark. It follows from this result that $(-1) \cdot a = -(1 \cdot a) = -a$.

4. $(-a) \cdot (-b) = a \cdot b$.

   Proof. First show that $-(-x) = x$ for all $x \in \mathbb{F}$. By definition, $x + (-x) = 0$, and $(-(-x)) + (-x) = 0$. By the uniqueness of the solution of the equation $y + (-x) = 0$, we have $x = -(-x)$. By (a), we have $(-a) \cdot (-b) = -((-a) \cdot b) = -(-(a \cdot b)) = a \cdot b$.

   Remark. It follows from this result that
   $-(a - b) = (-1) \cdot (a - b) = (-1) \cdot a + (-1) \cdot (-b) = -a + 1 \cdot b = b - a$.

5. $a/a = 1$ for all $a \in \mathbb{F} \setminus \{0\}$.

   Proof I. By definition, $a/b = a \cdot b^{-1}$, so we have $a/a = a \cdot a^{-1} = 1$.

   Proof II. By definition, a/a is the unique solution of the equation $a\cdot x = a$ in x. Rewrite the equation into $0 = a\cdot x+(-a) = a\cdot x+(-a)\cdot 1 = a\cdot x+ a\cdot(-1) = a\cdot(\ x+(-1)\ )$. It follows from that $a \neq 0$ and after multiplying the reciprocal of a, we have $x - 1 = 1 \cdot (x - 1) = (a^{-1} \cdot a) \cdot (x - 1) = a^{-1}\cdot(a\cdot(x-1)\ ) = a^{-1}\cdot 0 = 0$. And then $a/a = x = x+0 = x+(\ (-1)+1) = (x - 1) + 1 = 0 + 1 = 1$.

   Remark. Sometimes, one can start from the axiom of cancellation to solve equation: $a \cdot x = a \cdot b \ \wedge\ a \neq 0 \Rightarrow x = b$. In part, proof II is based on the cancellation.

6. $(a \cdot b)^{-1} = a^{-1} \cdot b^{-1}$ for all $a, b \in \mathbb{F} \setminus \{0\}$.

   Proof. As $(a \cdot b)^{-1}$ is the unique solution of the equation $(a \cdot b) \cdot x = 1$. It remains to show $a^{-1} \cdot b^{-1}$ also satisfies the equation above. From $1 = (a \cdot b) \cdot x = a \cdot (b \cdot x)$, So we have $a^{-1} = a^{-1} \cdot 1 = a^{-1} \cdot (\ a \cdot (b \cdot x)\ ) = (a^{-1} \cdot a) \cdot (b \cdot x) = 1 \cdot (b \cdot x) = b \cdot x$. Repeating the same procedure, we have $b^{-1} \cdot a^{-1} = b^{-1} \cdot (b \cdot x) = ... = x$.

   Remark. As $a \cdot a^{-1} = 1$, so $a^{-1} = 1/a$. And so it follows $\dfrac{1}{a \cdot b} = \dfrac{1}{a} \cdot \dfrac{1}{b}$.

7. $(\dfrac{a}{b}) \cdot (\dfrac{c}{d}) = \dfrac{a \cdot c}{b \cdot d}$ for all $b, d \in \mathbb{F} \setminus \{0\}$.

   Proof. We first to show that $\left(\dfrac{1}{a}\right) \cdot b = \dfrac{b}{a}$ by two methods:

   I. $\left(\dfrac{1}{a}\right) \cdot b = (1 \cdot a^{-1}) \cdot b = a^{-1} \cdot b = b/a$.

   II. Let $y = \dfrac{b}{a}$, then $ay = b$. Multiplying a^{-1}, we have $y = a^{-1} \cdot b = \dfrac{1}{a} \cdot b$.

   Returning to the proof of the original problem, $\dfrac{a}{b} \cdot \dfrac{c}{d} = (a \cdot \dfrac{1}{b}) \cdot (\dfrac{1}{d} \cdot c) = ... = a \cdot (\dfrac{1}{b} \cdot \dfrac{1}{d}) \cdot c = (a \cdot \dfrac{1}{b \cdot d}) \cdot c = ... = (a \cdot c) \cdot \dfrac{1}{b \cdot d} = \dfrac{a \cdot c}{b \cdot d}$.

8. $\dfrac{a}{b} = \dfrac{a \cdot c}{b \cdot c}$, where b and $c \neq 0$.

   Proof. Let x be the solution of the equation $bx = a$, then $x = \frac{a}{b}$, we have $(c \cdot b) \cdot x = c \cdot (b \cdot x) = c \cdot a$. By the uniqueness of the solution, we know that $\dfrac{a}{b} = x = \dfrac{c \cdot a}{c \cdot b}$.

9. $\dfrac{a}{c} + \dfrac{b}{c} = \dfrac{(a + b)}{c}$.

   Proof. Let $x = \dfrac{a}{c}$ and $y = \dfrac{b}{c}$. Then we have $c \cdot x = a$ and $c \cdot y = b$.

Adding these together, we have $c \cdot (x+y) = c \cdot x + c \cdot y = a + b$. By the uniqueness of the solution, we have $x + y = \dfrac{a+b}{c}$.

10. $\dfrac{(-a)}{b} = \dfrac{a}{-b} = -\left(\dfrac{a}{b}\right)$.

Proof I. $\dfrac{(-a)}{b} = \dfrac{(-a)\cdot(-1)}{b\cdot(-1)} = \dfrac{a}{-b}$. And $-\left(\frac{a}{b}\right) = -\left(a \cdot \frac{1}{b}\right) = (-a)\cdot\frac{1}{b} = \frac{-a}{b}$.

Proof II. $\dfrac{(-a)}{b} = \dfrac{1}{b}\cdot(-a) = -\left(\dfrac{1}{b}\cdot a\right) = -\left(\dfrac{a}{b}\right)$. Let $y = \dfrac{a}{-b}$, then we have $a = (-b)\cdot y = -(b \cdot y)$. So we have $a + (b\cdot y) = 0$. Hence we have $b \cdot y = b \cdot y + 0 = b\cdot y + (a + (-a)) = (b \cdot y + a) + (-a) = 0 + (-a) = -a$. By the uniqueness, we have $y = \dfrac{-a}{b}$.

11. $\dfrac{(a/b)}{(c/d)} = \dfrac{a\cdot d}{b \cdot c} = \left(\dfrac{a}{b}\right)\cdot\left(\dfrac{d}{c}\right)$, for all $b,\ c \in \mathbb{F}\setminus\{0\}$.

Proof. Let $y = \dfrac{(a/b)}{(c/d)}$. Then $\left(\dfrac{c}{d}\right)\cdot y = \dfrac{a}{b}$. After multiplying by $\dfrac{d}{c}$, we have $y = 1 \cdot y = \dfrac{dc}{cd}\cdot y = \left(\dfrac{d}{c}\cdot\dfrac{c}{d}\right)\cdot y = \dfrac{d}{c}\cdot\left(\dfrac{c}{d}\cdot y\right) = \dfrac{d}{c}\cdot\dfrac{a}{b} = \dfrac{ad}{bc} = \dfrac{a}{b}\cdot\dfrac{d}{c}$.

12. $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad+bc}{bd}$ for all $b, d \in \mathbb{F}\setminus\{0\}$.

Proof I. $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad}{bd} + \dfrac{cb}{bd} = \dfrac{ad+bc}{bd}$.

Proof II. Let $y = \dfrac{ad+bc}{bd}$, then $(bd)\cdot y = ad + bc$. Check that
$(bd)\cdot\left(\dfrac{a}{b}+\dfrac{c}{d}\right) = (bd)\cdot\dfrac{a}{b} + (bd)\cdot\dfrac{c}{d} = ... = \dfrac{b\cdot ad}{b\cdot 1} + \dfrac{d \cdot bc}{d\cdot 1} = \dfrac{b}{b}\cdot\dfrac{ad}{1} + \dfrac{d}{d}\cdot\dfrac{bc}{1} = ...$
$= 1\cdot(ad) + 1\cdot(bc) = ad + bc$. Thus by uniqueness of the solution of the equation $(bd)\cdot y = ad + bc$, we know that $\dfrac{ad+bc}{bd} = \dfrac{a}{b} + \dfrac{c}{d}$.

Remark. There are some proofs which are shorter, if one makes use of the existence of a^{-1} whenever $a \neq 0$.