EDUC 250 Mathematical Analysis Solution of Homework II

Due: 15th September, 2004. Hand in before the lecture starts at 9:00 a.m.

1. Suppose that a, b, c and d are in an ordered field $\mathbb{F}$. If $a > b$ and $c > d$, prove that

   (i) $a + c > b + d$; (ii) $a - b > d - c$;

   (iii) $-b + c > -a + d$; (iv) $-b - d > -a - c$.

   Solution. It follows from $a > b$ and $c > d$ that $(a-b)+(c-d) > 0+0 = 0$.

   (i) It follows from$(a + c) - (b + d) = (a - b) + (c - d) > 0$ that $a + c > b + d \Longleftrightarrow 0$.

   (ii) It follows from$(a-b)-(d-c) = (a-b)+(c-d) > 0$ that $a-b > d-c$.

   (iii) It follows from$(-b + c) - (-a + d) = (a - b) + (c - d) > 0$ that $-b + c > -a + d$.

   (iv) It follows from$(-b - d) - (-a - c) = (a - b) + (c - d) > 0$ that $-b - d > -a - c$.

2. Suppose that $a_1, a_2, \cdots, a_m,\ b_1, b_2, \cdots, b_n$ are elements in a ring $\mathbb{F}$, which is not necessarily commutative. Use the mathematical induction to prove that

$$\left(\sum_{i=1}^{m} a_i\right) \cdot \left(\sum_{j=1}^{n} b_j\right) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_i b_j\right).$$

   Hint: Apply mathematical induction on the sum $m + n$.

   Proof. Instead of using double indices m and n, we consider the sum $m + n$. Let $P(N)$ be the following statement involving a positive integer $N \geq 1$: $\left(\sum_{i=1}^{m} a_i\right) \cdot \left(\sum_{j=1}^{n} b_j\right) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_i b_j\right)$ holds, where $m, n \geq 1$ and $m + n = N + 1$. The proof proceeds by mathematical induction on the number N :

   (a) $N = 1$: then $(m, n) = (1, 1)$. Nothing to be prove in this case: $\left(\sum_{i=1}^{m} a_i\right) \cdot \left(\sum_{j=1}^{n} b_j\right) = (a_1) \cdot (b_1)$; and $\sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_i b_j\right) = (a_1 \cdot b_1)$.

   (b) Suppose that $P(N)$ holds for all $N \leq k$, we want to prove $P(k+1)$ holds as well. Then we divide into 2 cases:

      i. $(m, n) = (k, 1)$: From general distributive law (prove it!), we have

      L.H.S.$= \left(\sum_{i=1}^{m} a_i\right) \cdot b_1 = \left(\sum_{i=1}^{m} a_i \cdot b_1\right) = \sum_{i=1}^{m} \left(\sum_{j=1}^{1} a_i b_j\right) =$ R.H.S.

      ii. $n \geq 2$. So we have $n - 1 \geq 1$ and $m + (n - 1) = k$, so we can apply $P(k)$:

$$\begin{aligned}
\text{L.H.S.} &= \left(\sum_{i=1}^{m} a_i\right) \cdot \left(\sum_{j=1}^{n} b_j\right) \\
&= \left(\sum_{i=1}^{m} a_i\right) \cdot \left[\left(\sum_{j=1}^{n-1} b_j\right) + b_n\right] \\
&= \left(\sum_{i=1}^{m} a_i\right) \cdot \left(\sum_{j=1}^{n-1} b_j\right) + \left(\sum_{i=1}^{m} a_i\right) \cdot b_n \quad \text{(gen. dist. law)} \\
&= \sum_{i=1}^{m} \left(\sum_{j=1}^{n-1} a_i b_j\right) + \left(\sum_{i=1}^{m} a_i b_n\right) \quad \text{(MI and gen. dist. law)} \\
&= \sum_{i=1}^{m} \left[\left(\sum_{j=1}^{n-1} a_i b_j\right) + a_i b_n\right] \quad \text{(gen. comm. and asso. law)} \\
&= \text{R.H.S.}
\end{aligned}$$

Remark. State and prove the generalized commutative and associative laws for addition and generalized distributive law.