EDUC 250 Mathematical Analysis Homework III

Due: 21st September, 2004. Hand in before the lecture starts at 9:00 a.m.

1. Suppose that a, b and a_i, b_i ($i = 1, 2, \cdots, n$) are in an ordered field $\mathbb{F}$.

   (a) Prove that $\dfrac{|a| + a}{2} = \begin{cases} 0 & \text{if } a < 0; \\ a & \text{if } a \geq 0. \end{cases}$

   Proof. If $a < 0$, then $|a| = -a$, so we have $\dfrac{|a| + a}{2} = \dfrac{-a + a}{2} = \dfrac{0}{2} = 0$.
   If $a \geq 0$, then $|a| = a$, so we have $\dfrac{|a| + a}{2} = \dfrac{a + a}{2} = a$. The last equality follows from $a + a = 1 \cdot a + 1 \cdot a = (1 + 1) \cdot a = 2 \cdot a$.

   (b) Define $a \vee b = \max\{a, b\} = \begin{cases} a & \text{if } b < a; \\ b & \text{if } b \geq a, \end{cases}$ and

   $a \wedge b = \min\{a, b\} = \begin{cases} b & \text{if } b < a; \\ a & \text{if } b \geq a. \end{cases}$

   Prove that (i) $\dfrac{a + b + |b - a|}{2} = \max\{a, b\}$ and,
   (ii) $\dfrac{a + b - |b - a|}{2} = \min\{a, b\}$.

   Proof. (i) If $a \geq b$, then $a - b \geq 0$, and so $|b - a| = a - b$, in this case, $\dfrac{a + b + |b - a|}{2} = \dfrac{a + b + a - b}{2} = \dfrac{a + a}{2} = a = \max\{a, b\}$.
   Suppose that $a < b$, then $|b - a| = b - a$, and so $\dfrac{a + b + |b - a|}{2} = \dfrac{a + b + b - a}{=} b + b2 = b = \max\{a, b\}$.
   (ii) Leave it to you.

2. Prove, by means of mathematical induction, that

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + \cdots + |a_n|.$$

   Determine when the equality holds.

   Proof. Proceed by means of mathematical induction on $n \geq 1$:

   (a) If $n = 1$ then $|a_1| = |a_1|$, and if $n = 2$, then it follows from triangle inequality.

   (b) Suppose that the inequality holds for $n = 1, 2, \cdots, k$, then we now prove the inequality holds for $n = k + 1$. Let a_i $(1 \leq i \leq k + 1)$ be in $\mathbb{F}$, then $|a_1 + a_2 + \cdots + a_{k+1}| = |(a_1 + a_2 + \cdots + a_k) + a_{k+1}| \leq |a_1 + a_2 + \cdots + a_k| + |a_{k+1}| \leq (|a_1| + \cdots + |a_k|) + |a_{k+1}|$.

   And hence result follows from mathematical induction.
   Return to the cases of equality, it holds if and only if the signs of $a_1, a_2, \cdots, a_n$ are all non-negative, or all all non-positive. We leave the proof as an exercise.

3. Let $S = \{ a_1, a_2 \cdots, a_n \}$ be a set of n elements in $\mathbb{F}$. Define $\max\{a_1, a_2, \cdots, a_n\} = \max\{ \max\{a_1, a_2, \cdots, a_{n-1}\}, a_n \}$ for $n \geq 3$ inductively, and $\min S$ is defined similarly. Prove that $\max S$ and $\min S$ is independent of the choice of the ordering of the elements, i.e.$\max\{ a_{\pi(1)}, a_{\pi(2)} \cdots, a_{\pi(n)} \} = \max\{a_1, a_2, \cdots, a_n\}$, where π is a permutation of the index set $\{1, 2, \cdots, n\}$, and similar equality for $\min$.
   Proof. Let $\pi : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$ be a bijective mapping. Let $B_k = \max\{ a_{\pi(1)}, a_{\pi(2)} \cdots, a_{\pi(k)} \}$, and $A_k = \max\{a_1, a_2, \cdots, a_k\}$ for $1 \leq k \leq n$. Want to prove that $B_n = A_n$.

   (a) By definition of max, we have $B_n \geq a_{\pi(n)}$ and $B_n \geq B_{n-1}$, and it must be equal to one of them. It follows from mathematical induction that $B_n \geq a_{\pi(i)}$ $(1 \leq i \leq n)$, and B_n must be equal to one of them, says $a_{\pi(I)}$.

   (b) Similarly, we have $A_n \geq a_i$ $(1 \leq i \leq n)$, and A_n must be equal to one of them, says a_J. Then let K such that $\pi(K) = J$.

   Then $A_n = a_J = a_{\pi(K)} \leq B_n$, and similarly, $B_n = a_{\pi(I)} \leq A_n$. Thus $A_n = B_n$.

4. Suppose that $b_i > 0$ ($i = 1, 2, \cdots, n$), and $S = \{ \frac{a_1}{b_1}, \frac{a_2}{b_2}, \cdots, \frac{a_n}{b_n} \}$, prove, by means of mathematical induction on n, that

$$\min S \le \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \le \max S.$$

Proof. Let $\max S = M$, then we have $\frac{a_i}{b_i} \le M$, and hence $a_i \le b_i M$ $(i = 1, 2, \cdots, n)$. Adding these inequalities (in fact it follows from mathematical induction on n, which we had omitted here), we have $a_1 + a_2 + \cdots + a_n \le (b_1 M + b_2 M + \cdots + b_n M) = M \cdot (b_1 + b_2 + \cdots + b_n)$. As $b_1 + b_2 + \cdots + b_n > 0 + 0 + \cdots + 0 = 0$, we have $\frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \le M$. One can prove the similar result for $\min S$.

5. (Supplementary Question). Let $\mathbb{F}$ be an ordered field. Prove that (i) $x^2 + xy + y^2 \ge 0$ for any $x, y \in \mathbb{F}$; (ii) Equality holds if and only if $x = y = 0$.
Proof. One can easily prove that $2 = 1 + 1 > 0, 3 = 2 + 1 > 0 + 0 = 0$ and $4 = 3 + 1 > 0 + 0 = 0$. Moreover, we know that $z^2 \ge 0$ for all $z \in \mathbb{F}$. So we have $4(x^2 + xy + y^2) = (x + x + y)^2 + (y^2 + y^2 + y^2) \ge 0 + 0 = 0$. Then the result follows from $4 > 0$ and division by 4.
Equality holds if and only if $(2x + y)^2 = 0$ and $y^2 + y^2 + y^2 = 0$, which in turns are equivalent to $x = y = 0$.

6. (Supplementary Question). Prove that: if $a \ne 0$ and n is a positive integer, then $\underbrace{a + a + \cdots + a}_{n \text{ terms}} \ne 0$.
Proof. It suffice to prove that $\underbrace{1 + 1 + \cdots + 1}_{n \text{ terms}} \ne 0$. We know that $1 \in P$, and it follows from the positivity is closed under addition and mathematical induction that $\underbrace{1 + 1 + \cdots + 1}_{n \text{ terms}} \in P$, and hence the sum is non-zero.
Remark. Even though, we have not introduced natural numbers yet.