EDUC 250 Mathematical Analysis Solution of Homework IV

Due: 22nd September, 2004. Hand in before the lecture starts at 9:00 a.m.

Let $A, B$ be non-empty, bounded subset of an ordered field $\mathbb{F}$.

1. Prove that $\inf A \geq \inf B$ if $A \subset B$.

   **Proof.** As $\inf B$ is an lower bound of $B$, we have $b \geq \inf B$ for all $b \in B$. Since $A \subset B$, then $a \geq \inf B$ for all $a \in A$. In particular, $\inf B$ is a lower bound of the set $A$. Finally, $\inf A$ is the greatest lower bound of $A$, so we have $\inf A \geq \inf B$.

2. Define $A + B = \{ \ a + b \in \mathbb{F} \mid a \in A, \ b \in B \ \}$. Prove that

$$\inf(A + B) \geq \inf A + \inf B.$$

   **Proof.** As $\inf A \leq a$ and $\inf B \leq b$ for all $a \in A$ and $b \in B$, it follows that $\inf A + \inf B \leq a + b$. Hence by definition of $A + B$, we know that $\inf A + \inf B$ is a lower bound of $A + B$. By definition of inf, we know that $\inf A + \inf B \leq \inf(A + B)$.

3. Define $-A = \{ \ -x \in \mathbb{F} \mid x \in A \ \}$. Prove that

   (i) $\sup(-A) = -\inf A$, and (ii) $\inf(-A) = -\sup A$.

   **Proof.** (ii) follows from (i) by replacing $A$ by $-A$ and that $-(-A) = A$.

   (a) For any upper bound $s$ of $-A$, we have $x \leq s$ for all $x \in -A$, it follows that $-x \in A$. Thus we have $-x \geq -s$, and so $-s$ is a lower bound of $A$. Then by definition of inf, we know that $\inf A \geq -s$ for all upper bound $s$ of $-A$. In particular, $\inf A \geq -\sup(-A)$, i.e. $-\inf A \leq \sup(-A)$.

   (b) Now we will prove that $-\inf A \geq \sup(-A)$ : As $\inf A$ is a lower bound of $A$, we have $\inf A \leq a$ for all $a \in A$. hence $-\inf A \geq -a$ for all $-a \in -A$. In particular, $-\inf A$ is an upper bond of $-A$, and it follows from the definition of sup we have $-\inf A \geq \sup(-A)$.

4. (Supplementary problem).

   (i) Prove that $\sup( \ A + (-A) \ ) \geq 0$.

   (ii) Determine when the equality holds.

   (iii) How do you modify if you replace sup by inf .

   **Proof.** (i) Since $A \neq \emptyset$, there exists $a \in A$, and hence $-a \in -A$. In particular, $0 = (a) + (-a) \in A + (-A)$, so $\sup A \geq 0$.

   (ii) $\sup(A + (-A) \ ) = 0$ if and only if $A = \{a\}$ for some $a \in \mathbb{F}$.

   ($\Leftarrow$) If $A = \{a\}$, then $A + (-A) = \{0\}$, so $\sup(A + (-A)) = 0$.

   ($\Rightarrow$) Suppose contrary, then $A$ contains at least two elements $a, b \in \mathbb{F}$ ($a \neq b$). Then $-a, -b \in -A$, so $a \pm b, b \pm a \in A + (-A)$. It follows from the definition of upper bound, we have $a \pm b \leq 0$ and $b \pm a \leq 0$. In particular, $a \leq b$ and $b \leq a$. In particular, $a = b$ which violates the assumption that $a \neq b$.

   (iii) It follows from $0 \in A + (-A)$ that $\inf(A + (-A) \ ) \leq 0$. And $\inf(A + (-A) \ ) = 0$ if and only if $A = \{a\}$. The proof follows from question 3, and $-( \ A + (-A) \ ) = A + (-A)$.