EDUC 250 Mathematical Analysis   Solution of Homework V

Due: 28th September, 2004. Hand in before the lecture starts at 9:00 a.m.

1. (Very Important) Prove that if $\varepsilon > 0$, then there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \varepsilon$.

   Proof. For any $\varepsilon > 0$, then $\frac{1}{\varepsilon} > 0$, then by Archimedean property, there exists $n \in \mathbb{N}$ such that $(0 <)\frac{1}{\varepsilon} < n$. As $\varepsilon > 0$ and $n > 0$, after multiplying $\dfrac{\varepsilon}{n} > 0$ we have $\dfrac{1}{n} = \dfrac{\varepsilon}{n} \cdot \dfrac{1}{\varepsilon} < \dfrac{\varepsilon}{n} \cdot n = \varepsilon$.

2. Let S be a non-empty subset of $\mathbb{R}$. Show that $s = \sup S$ if and only if the following two conditions hold:

   (a) for every positive integer n, $s - 1/n$ is not an upper bound of S;

   (b) for every positive integer n, $s + 1/n$ is an upper bound of S.

   Proof. ($\Rightarrow$) Suppose that $s = \sup S$, then s is an upper bound of S, in particular, $s \leq s + 1/n$ is also an upper bound of S for all $n \geq 1$. Hence (b) holds. Now we will prove that (a) holds. For any $n \in \mathbb{N}$ we have $\frac{1}{n} > 0$, then it follows from equivalent definition of supremum that $x > s - \frac{1}{n}$. In particular, $s - \frac{1}{n}$ is not an upper bound of S.

   ($\Leftarrow$) Suppose that s satisfies the conditions (a) and (b).

   (i) We first prove that s is an upper bound of S. Suppose contrary, then there exists $x \in S$ such that $x > s$. It follows from question 1 that there exists $\frac{1}{n} < x - s$. In particular, $x > s + \frac{1}{n}$, which violates condition (b).

   (ii) We then prove that $s = \sup S$. For any $\varepsilon > 0$, it follows from Archimedean property, there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \varepsilon$. Then by (a), we have $s - \frac{1}{n}$ is not an upper bound of S, i.e. there exists $x \in S$ such that $s - \frac{1}{n} < x$. Hence we have $s - \varepsilon < s - \frac{1}{n} < x$. It follows from the equivalent definition of supremum that $s = \sup S$.

3. Let $\emptyset \neq S$ $(\subset \mathbb{R})$ be bounded. Let $aS = \{ \ as \mid s \in S \ \}$ for any $a \in \mathbb{R}$.

   (a) If $a > 0$, prove that $\inf(aS) = a \inf S$ and $\sup(aS) = a \sup S$.

   (b) If $a < 0$, prove that $\sup(aS) = a \inf S$ and $\inf(aS) = a \sup S$.

   Proof I. We just give a proof of (a), and leave the proof (b) to you.

   (i) For any $x \in S$ we have $x \leq \sup S$. Then multiplying by a, we have $ax \leq a \sup S$ for all $x \in S$, hence $a \sup S$ is an upper bound of aS.

   (ii) For any $\varepsilon > 0$, we have $\varepsilon / a > 0$. Then by equivalent definition of supremum, there exists $x \in S$ such that $x > \sup S - \varepsilon/a$. Multiplying by $a > 0$, $a \cdot x > a \cdot \sup S - \varepsilon$. Consequently, $\sup(aS) = a \sup S$.

   (iii) Replacing A by $-A$, we have $a \cdot (-A) = -(a \cdot A)$. Then the result $\inf(aS) = a \inf S$ follows from $\inf(-A) = -\sup A$.

   Proof II. Assume that $a > 0$. (i) For any $y \in aS$, there exists $x \in S$ such that $y = ax$. By definition of supremum, we have $y = ax \leq a \cdot \sup S$. Hence $a \cdot \sup S$ is an upper bound of the set aS. By principle of Supremum, $\sup(aS)$ exists in $\mathbb{R}$. In particular, $\sup(aS) \leq a \sup S$.

   (ii) It remains to show that $\sup(aS) \geq a \sup S$. It is easy to check that $(1/a) \cdot (a \sup S) = S$. By replacing S and a by aS and $1/a$ respectively. Hence it follows from (i) that $\sup(S) = \sup(\ (1/a)(aS) \ ) \leq (1/a) \sup(aS)$. In particular, $\sup(aS) \geq a \sup S$.

4. Let A and B be two non-empty, bounded subset of $\mathbb{F}$. Prove that $\sup(A + B) = \sup A + \sup B$.

   Hint: Instead of ε, break it into two $\varepsilon/2$.

   Proof. We had established $\sup(A+B) \leq \sup A + \sup B$ in the class, and hence we know that $\sup A + \sup B$ is an upper bound of the set $A + B$. It remains to show that $\sup A + \sup B$ is the least upper bound of the set $A + B$. Then for any $\varepsilon > 0$, there exist element $a \in A$ and $b \in B$ such that the followings hold: $\quad a \geq \sup A - \varepsilon/2$, and $b \geq \sup B - \varepsilon/2$.

   Then the element $x = a + b \in A + B$ satisfies the following:

$x = a + b \geq (\sup A - \varepsilon/2) + (\sup B - \varepsilon/2) = (\sup A + \sup B) - \varepsilon$. Then the result follows from equivalent definition of sup.

5. Let A and B be two non-empty, bounded subsets of an ordered field $\mathbb{F}$, define $A \cdot B = \{a \cdot b \mid a \in A, b \in B\}$.

   (a) Prove $A \cdot B$ is bounded;

   (b) Prove that $\sup(A \cdot B)$
   $= \max\{\sup A \cdot \sup B,\ \sup A \cdot \inf B,\ \inf A \cdot \sup B,\ \inf A \cdot \inf B\}$.

   (c) If we replace "bounded" by "bounded above", is $A \cdot B$ bounded above? Justify your answer.

Hint: For (b), in case of $\sup(A \cdot B) = \sup A \cdot \inf B$ one can consider the following setup: For any $\varepsilon > 0$, there exists $x \in A$ and $y \in B$ such that $x \geq \sup A - \varepsilon_1$ and $y \geq \inf B + \varepsilon_2$, where $\varepsilon_1 = \dfrac{\varepsilon}{2(|\inf B| + 1)}$ and $\varepsilon_2 = \min\{\dfrac{\varepsilon}{3(|\sup A| + 1)}, \dfrac{1}{6}\}$.

Proof. (a) Since A, B are bounded, by the supremum principle and similar version for infimum, all the infimum and supremum involved exist in $\mathbb{R}$, $\inf A \leq a \leq \sup A$, and $\inf B \leq b \leq \sup B$ for all $a \in A$ and all $b \in B$. Then we have $a \cdot b \leq \sup A \sup B$ if $a \geq 0$ and $b \geq 0$. But if there are some negative elements in either A or B, then one need to modify the direction of the inequalities when we estimate the product ab with those bounds, so we have $ab \leq \max\{\sup A \cdot \sup B,\ \sup A \cdot \inf B,\ \inf A \cdot \sup B,\ \inf A \cdot \inf B\}$. It follows that $A \cdot B$ is bounded above. Similarly, one know that $ab \geq \min\{\sup A \cdot \sup B,\ \sup A \cdot \inf B,\ \inf A \cdot \sup B,\ \inf A \cdot \inf B\}$.

(b) There are 4 cases depending on which one is $\max\{\sup A \cdot \sup B,\ \sup A \cdot \inf B,\ \inf A \cdot \sup B,\ \inf A \cdot \inf B\}$. We only prove the case that $\sup A \cdot \inf B$ is the largest among these four products, for example, $B = \{1, 2\}$ and $A = \{-1, -2\}$ then $A \cdot B = \{-1, -2, -4\}$. Hence $\sup(A \cdot B) = -1$ and $\sup A \cdot \inf B = (-1) \cdot 1 = -1$.

- As $\sup A \cdot \inf B \geq \sup A \cdot \sup B$, we have $\sup A \cdot (\sup B - \inf B) \leq 0$. Similarly, it follows from $\sup A \cdot \inf B \geq \inf A \cdot \inf B$, we know that $\inf B \cdot (\sup A - \inf A) \geq 0$.
- Without loss of generality, we may assume that $\sup B > \inf B$ and $\sup A > \inf A$ otherwise $B = \{b\}$ or $A = \{a\}$ for some $a, b \in \mathbb{R}$. Then the resulting equality easily follows from question 2.
- It follows that $\sup A \leq 0$, and that $\inf B \geq 0$. Hence $a \leq 0$ for all $a \in A$, and $0 \leq \inf B \leq b$ for all $b \in B$.

  After multiplying $b \geq \inf B$ by a, we have $ab \leq \sup A \cdot \inf B$. Hence $\sup A \cdot \inf B$ is an upper bound of the set $A \cdot B$.
- It remains to show that $\sup A \cdot \inf B = \sup(A \cdot B)$.
  For any $\varepsilon > 0$, there exists $x \in A$ and $y \in B$ such that $x \geq \sup A - \varepsilon_1$ and $y \leq \inf B + \varepsilon_2$, where $\varepsilon_1 = \dfrac{\varepsilon}{2(|\inf B| + 1)}$ and $\varepsilon_2 = \min\{\dfrac{\varepsilon}{3(|\sup A| + 1)}, \dfrac{1}{6}\}$. Then it follows from $x \leq 0$ and $\inf B \geq 0$ that $x \cdot y \geq x \cdot (\inf B + \varepsilon_2)$

$$
\begin{aligned}
&\geq (\sup A - \varepsilon_1) \cdot (\inf B + \varepsilon_2) \\
&= \sup A \cdot \inf B + (\sup A \cdot \varepsilon_2 - \inf B \cdot \varepsilon_1) - \varepsilon_1 \varepsilon_2 \\
&\geq \sup A \cdot \inf B + \left(\frac{-|\sup A|}{2(|\sup A| + 1)} \cdot \varepsilon - \frac{|\inf B|}{3(|\inf B| + 1)} \cdot \varepsilon \right) - \varepsilon \cdot \frac{1}{6} \\
&\geq \sup A \cdot \inf B - \frac{\varepsilon}{2} - \frac{\varepsilon}{3} - \frac{\varepsilon}{6} \\
&= \sup A \cdot \inf B - \varepsilon.
\end{aligned}
$$

Consequently, $\sup A \cdot \inf B = \sup(A \cdot B)$.

Remark. It is easier to separate the set A into 2 parts: $A = A_+ \cup A_-$, where $A_+ = \{a \in A \mid a \geq 0\}$ and $A_- = \{a \in A \mid a < 0\}$. Define B_+ and B_- similarly. If any of these subsets are empty, we don't need to write them down in the following.

Then it remains to show that
$A \cdot B = (A_+ \cdot B_+) \cup (A_+ \cdot B_-) \cup (A_- \cdot B_+) \cup (A_- \cdot B_-)$. And It follows that

$\sup(A \cdot B) = \max\{ \sup(A_+B_+), \sup(A_+B_-), \sup(A_-B_+), \sup(A_-B_-) \}$

$= \max\{ \sup(A_+)\sup(B_+),\ \sup(A_+)\inf(B_-),\ \sup(A_-B_+),\ \inf(A_-)\inf(B_-) \}.$

Study these solution carefully.

$0 \le \inf A_+ \le x \le \sup A_+$ for all $x \in A_+$, and $\inf B_- \le y \le \sup B_- \le 0$ for all $y \in B_-$. Hence we have $0 \ge xy \ge y \sup A_+ \ge \inf B_- \sup A_+$, and $xy \le y \inf A_+ \le \sup B_- \inf A_+$. Thus $\sup(A_+ \cdot B_-) \le \inf A_+ \sup B_-$.
It remains to show that $\sup(A_+ \cdot B_-) = \inf A_+ \sup B_-$.

Choose any positive number $M > 1$ such that $|\inf A_+| \le M/2$, and $|\sup B_-| \le M/2$. For any given $\varepsilon > 0$ there exist $x \in A_+$ and $y \in B_-$ such that $x < \inf A_+ + \varepsilon_1/M$ and $y > \sup B_- - \varepsilon_2/M$, where ε_i $(i = 1, 2)$ are to be chosen later.

Since $x \ge 0$ and $y < 0$, Then $\quad x \cdot y$

$\ge x \cdot (\sup B_- - \varepsilon_2/M)$

$> (\inf A_+ + \varepsilon_1/M) \cdot (\sup B_- - \varepsilon_2/M)$

$= \inf A_+ \sup B_- - \dfrac{|\sup B_-|}{M}\varepsilon_1 - \dfrac{\inf A_+}{M}\varepsilon_2 - \dfrac{\varepsilon_1\varepsilon_2}{M^2}$

$> \inf A_+ \sup B_- - \dfrac{1}{2}\varepsilon_1 - \dfrac{1}{2}\varepsilon_2 - \varepsilon_1\varepsilon_2$

$\ge \inf A_+ \sup B_- - \dfrac{1}{2}\varepsilon - \dfrac{1}{2} \cdot \dfrac{\varepsilon}{3} - \varepsilon \cdot \dfrac{1}{3} = \inf A_+ \sup B_- - \varepsilon.$

In the last step, we had chosen $\varepsilon_1 = \varepsilon$ and $\varepsilon_2 = \min\{1, \varepsilon\}/3$.

(c) The result (a) does not hold if we replace "bounded" by "bounded above". Let $A = B = (-\infty, 0)$, then $A \cdot B = (0, +\infty)$ which is **not** bounded above.