EDUC 250 Mathematical Analysis Homework V

Due: 28th September, 2004. Hand in before the lecture starts at 9:00 a.m.

1. Prove that if $\varepsilon > 0$, then there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \varepsilon$.
2. Let $\emptyset \neq S\ (\subset \mathbb{R})$ be bounded. Let $aS = \{\ as \mid s \in S\ \}$ for any $a \in \mathbb{R}$.
   (a) If $a > 0$, prove that $\inf(aS) = a \inf S$ and $\sup(aS) = a \sup S$.
   (b) If $a < 0$, prove that $\sup(aS) = a \inf S$ and $\inf(aS) = a \sup S$.
3. Let S be a non-empty subset of $\mathbb{R}$. Show that $u = \sup S$ if and only if for every positive integer n, the number $u - 1/n$ is not an upper bound of S but the number $u + 1/n$ is an upper bound of S.
4. Let A and B be two non-empty, bounded subset of $\mathbb{F}$. Prove that $\sup(A + B) = \sup A + \sup B$.
5. Let A and B be two non-empty, bounded subsets of an ordered field $\mathbb{F}$, define $A \cdot B = \{a \cdot b \mid a \in A, b \in B\ \}$.
   (a) Prove $A \cdot B$ is bounded;
   (b) (Hard) Prove that $\sup(A \cdot B)$
   $= \max\{\ \sup A \cdot \sup B,\ \sup A \cdot \inf B,\ \inf A \cdot \sup B,\ \inf A \cdot \inf B\ \}$.
   (c) If we replace "bounded" by "bounded above", is $A \cdot B$ bounded above? Justify your answer.

   Hint: It would be easier if you assume that the sets A, B consists of non-negative numbers only.
   In general for part (b), in case of $\sup(A \cdot B) = \sup A \cdot \inf B$ one can consider the following setup: For any $\varepsilon > 0$, there exists $x \in A$ and $y \in B$ such that $x \geq \sup A - \varepsilon_1$ and $y \geq \inf B + \varepsilon_2$, where $\varepsilon_1 = \dfrac{\varepsilon}{2(|\inf B| + 1)}$ and $\varepsilon_2 = \dfrac{\varepsilon}{2(|\sup A| + 1)}$.

EDUC 250 Mathematical Analysis Homework VI

Due: 29th September, 2004. Hand in before the lecture starts at 9:00 a.m.

1. Let A and B be two non-empty, bounded subsets of the set P of all positive elements in an ordered field $\mathbb{F}$. If $0 \notin B$, define
   $$A/B = \{a/b \mid a \in A, b \in B\ \}.$$
   If $\inf B > 0$, prove that $\sup(A/B) = (\sup A)/(\inf B)$.

   Hint: (i) one can easily prove that $(\sup A)/(\inf B)$ is an upper bound of the set A/B by means of the definition of sup and inf.
   (ii) As for the equality, one can by replace ε by $\varepsilon \sup A > 0$ and $\varepsilon \inf B > 0$ respectively in the equivalent definition of $\sup(A/B)$.
2. Determine the supremum and infimum of the set
   $$A = \{\ \frac{m+n}{m^2 + mn + n^2} \in \mathbb{R} \mid m, n \in \mathbb{N}\ \}.$$

1 Supplementary Homework (No need to hand in)

1. Define $f : \mathbb{R} \to \mathbb{R}$ to be $f(x) = \dfrac{x^2}{1 + x^2}$, for all $x \in \mathbb{R}$.
   (a) Determine the range of f i.e. the set $f[\mathbb{R}]$.
   (b) Find the supremum and the infinum of the set $f[\mathbb{R}]$.
   (c) Find the supremum and the infinum of the set $f[\mathbb{N}]$.
2. Prove that (i) $\sup(A \cap B) = \min\{\ \sup A,\ \sup B\ \}$, and
   (ii) $\sup(A \cup B) = \max\{\ \sup A,\ \sup B\ \}$.
3. Let S be a nonempty set of $\mathbb{R}$ that is bounded from above, and let $\varepsilon > 0$. Prove that the interval $(\ \sup S - \varepsilon,\ \sup S\ ]$ contains an element of S. Does the converse hold? Does it holds if we replace the interval by the open interval $(\ \sup S - \varepsilon,\ \sup S\ )$?

2 Solution of Supplementary Homework

No need to hand in.

1. Define $f : \mathbb{R} \to \mathbb{R}$ to be $f(x) = \dfrac{x^2}{1+x^2}$, for all $x \in \mathbb{R}$.

   (a) Determine the range of f i.e. the set $f[\mathbb{R}]$.

   (b) Find the supremum and the infinum of the set $f[\mathbb{R}]$.

   (c) Find the supremum and the infinum of the set $f[\mathbb{N}]$.

   Proof.

   (a) $f[\mathbb{R}] = [0, 1)$. As $0 \le x^2 < x^2 + 1$, so we have $f(0) = 0 \le \frac{x^2}{x^2+1}$. Moreover, $f(x) = \frac{x^2}{x^2+1} < \frac{x^2+1}{x^2+1} = 1$. Hence $f[\mathbb{R}] \subset [0, 1)$. For any $t \in [0, 1)$, let $x = \sqrt{\frac{t}{1-t}} \in \mathbb{R}$, we have $f(x) = t$. Hence $f[\mathbb{R}] = [0, 1)$.

   (b) As $f[\mathbb{R}] = [0, 1)$, so $\sup f[\mathbb{R}] = 1$, and $\inf f[\mathbb{R}] = 0$.
   (Try to prove that $\sup(a, b) = b$ and $\inf(a, b) = a$, provided $a < b$.)

   (c) As $f(0) = 0$, we have $\inf f[\mathbb{N}] = 0$. It remains to show $\sup f[\mathbb{N}] = 1$. It follows from $\mathbb{N} \subset \mathbb{R}$ that $f[\mathbb{N}] \subset f[\mathbb{R}]$, and hence $\sup f[\mathbb{N}] \le \sup f[\mathbb{R}] = 1$. On the other hand, for any $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ such that $0 < 1/n < \varepsilon$, then
   $f(n) = 1 - \dfrac{1}{n^2+1} > 1 - \dfrac{1}{n^2} \ge 1 - \dfrac{1}{n} > 1 - \varepsilon$. So $\sup f[\mathbb{N}] = 1$.

2. Suppose that A and B are bounded above, non-empty subsets of an ordered field $\mathbb{F}$. Prove that

   (i) $\sup(A \cup B) = \max\{ \sup A, \ \sup B \}$, and

   (ii) $\sup(A \cap B) \le \min\{ \sup A, \ \sup B \}$ provided $A \cap B$ is non-empty. Does the equality hold?

   Proof. (i) As A and B are subset of $A \cup B$, we have $\sup A, \sup B \le \sup(A \cup B)$. Hence $\max\{\sup A, \ \sup B\} \le \sup(A \cup B)$.
   For any $x \in A \cup B$, we have $x \in A$ or $x \in B$, then $x \le \sup A$ or $x \le \sup B$, hence $x \le \max\{\sup A, \ \sup B\}$. So $\max\{\sup A, \ \sup B\}$ is an upper bound of union $A \cup B$, in particular, we have $\sup(A \cup B) \le \sup A$. Consequently, we have $\sup A = \sup(A \cup B)$.
   (ii) As $A \cap B$ is a non-empty subset of A and B. We have $\sup(A \cap B) \le \sup A$ and $\sup B$. It follows that $\sup(A \cap B) \le \min\{ \sup A, \ \sup B \}$.

   The equality *does not hold* in general. Let $A = [0, 1] \cap \mathbb{Q}$ and $B = [0, 1] \setminus A$. As $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$, we know that $\sup A = \sup B = 1$. However, $A \cap B = \emptyset$, which does not have any supremum.

3. (i) Let S be a nonempty, bounded above subset of $\mathbb{R}$, and let $\varepsilon > 0$. Prove that the interval $(\ \sup S - \varepsilon, \ \sup S \ ]$ contains at least one element of S.

   (ii) Does the converse hold?

   (iii) Does it holds if we replace the interval by $(\ \sup S - \varepsilon, \ \sup S \ )$?

   Proof. (i) Let $a = \sup S$, then a is the least upper bound of S, so $a - \varepsilon$ is not an upper bound of S. So there exists an element $x \in S$ with $x > a - \varepsilon$. As a is an upper bound of S, so $x \le a$. Thus $x \in S \cap (a - \varepsilon, a]$.
   (ii) The converse does not hold if we only assume that $(s - \varepsilon, s] \cap S$ is non-empty for any $\varepsilon > 0$, as s may not be an upper bound. This (geometric) assumption only corresponds to the condition (ii) of the equivalent definition of sup.
   (iii) The result does not hold, if we replace the interval by $(\ \sup S - \varepsilon, \ \sup S \ )$, for example, $S = \{0\}$.