EDUC 250 Mathematical Analysis Homework VII

Due: 6th October, 2004. Hand in before the lecture starts at 9:00 a.m.

1. Show that a finite subset S in $\mathbb{R}$ has no accumulation points.

2. Prove that every point of $I = [0, 1]$ is an accumulation point of $I \cap \mathbb{Q}$ and $I \setminus \mathbb{Q}$ respectively.

3. Let $a_n = \dfrac{3n^2 + 4}{n^2 + 8n + 7}$. State carefully any theorems you use about limits, prove that a_n converges to 3.
   Hint: Use $\left|\dfrac{3n^2 + 4}{n^2 + 8n + 7} - 3\right| \le \cdots \le \dfrac{24}{n}$.

4. (a) Determine the least upper bound of the set $S = \{\, 1 - \frac{1}{n} \mid n \in \mathbb{N} \,\}$.
   (b) What is the greatest lower bound of S?
   (c) Determine the set S^a of accumulation points of S.

5. Find the set of accumulation points of the set $\{\, 3(1-\frac{1}{n})+2(-1)^n \mid n \in \mathbb{N} \,\}$. Justify your answers.

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Due: 12th October, 2004. Hand in before the lecture starts at 9:00 a.m.

1. Let $K = \{\, 1/n \in \mathbb{R} \mid n = 1, 2, \cdots \,\} \cup \{0\}$. Prove that K is compact directly from the definition, without using Heine-Borel theorem.
   Hint: Consider the open interval containing the point 0. Recall that
   (a) $S \subset \mathbb{R}$ is called *compact* if for each family $\mathcal{F}$ of open covering of S, there exists a finite subfamily $\mathcal{F}_0$ of $\mathcal{F}$ such that $\mathcal{F}_0$ covers S.
   (b) **Heine-Borel Theorem**. Finite closed interval $[a, b]$ is a compact set, where $a < b$ are finite numbers.

2. Determine the sets A^a and B^a of accumulation points of $A = \{\, \dfrac{1}{n} + \dfrac{1}{m} \mid n, m \in \mathbb{N} \,\}$ and $B = \{\, \dfrac{m}{nm + 1} \mid n, m \in \mathbb{N} \,\}$ respectively.
   Hint: First fix m, and consider the limit of the sequence $\{\, \frac{1}{n} + \frac{1}{m} \mid n \ge 1\}$. Rewrite $\frac{m}{nm+1} = \frac{1}{n+1/m}$.

3. Let $S \subset \mathbb{R}$. A point $x \in \mathbb{R}$ is called an *adherent point* of S if for any $\varepsilon > 0$, the interval $(x - \varepsilon, x + \varepsilon) \cap S \ne \emptyset$. If S is bounded, show that $\sup E$ is an adherent point of E, and is also an adherent point of $\mathbb{R} \setminus E$.
   Recall that: A number $x \in \mathbb{R}$ is called an *accumulation point* (or *limit point*, *cluster point*) of S is for any $\delta > 0$, $(x - \delta,\ x + \delta) \cap S \setminus \{x\} \ne \phi$. This means that any punctured interval centered at x contains *at least* a point in S.

4. Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence of real numbers, define $x_n = \sup\{a_n, a_{n+1}, \cdots\}$ and $y_n = \inf\{a_n, a_{n+1}, \cdots\}$ for all $n \ge 1$.
   (a) Prove that (i) $(x_n)_{n\in\mathbb{N}}$ is monotone decreasing, and (ii) $(y_n)_{n\in\mathbb{N}}$ is monotone increasing.
   (b) Prove that (i) $\lim\limits_{n\to\infty} x_n$ and $\lim\limits_{n\to\infty} y_n$ exist; (ii) $\lim\limits_{n\to\infty} x_n \ge \lim\limits_{n\to\infty} y_n$.
   (c) The set $\{a_1, a_2, \cdots\}^a \subset [\, \lim\limits_{n\to\infty} x_n,\ \lim\limits_{n\to\infty} y_n \,]$.