2. **Definition**. Given $a, b \in \mathbb{F}$, define $a < b$ if $b - a \in P$. In this case, $<$ is a relation on the set $\mathbb{F}$. An element $x \in \mathbb{F}$ is called *positive* (*negative*), if $x \in P$ ($x \in -P$) respectively.

3. We also write $x > y$ to represent $y < x$, and we write $x \leq y \iff (x = y)$ or $(x < y)$. Similarly, $x \geq y \iff (x = y)$ or $(y < x)$. One can easily write down the analogy inequalities with $\leq$ instead of $<$.

3.4 Properties of Inequalities

For all $x, y, z \in \mathbb{F}$, we have

I-1 Transitivity: If $x < y$ and $y < z$, then $x < z$.
Proof. $(x < y) \wedge (y < z) \iff (y - x \in P) \wedge (z - y \in P) \implies z - x = (z - y) + (y - x) \in P \iff x < z$.

I-2 Trichotomy: Exactly one of the following holds: $x < y$, $x = y$, $x > y$.

I-3 If $x < y$, then $x + z < y + z$, for all $z \in \mathbb{F}$.
Proof. $x < y \iff 0 < y - x = (y + z) - (x + z) \iff x + z < y + z$.
It follows that $x + z < y + z$ is equivalent $x < y$.

I-4 (i) If $x < y$ and $z > 0$ then $x \cdot z < y \cdot z$.
(ii) If $x < y$ and $z < 0$ then $x \cdot z > y \cdot z$.
Proof. (i) We know that $y - x \in P$ and $z \in P$ then
$z \cdot y - (z \cdot y) = z \cdot (y - x) \in P$. This means that $z \cdot x < z \cdot y$.
(ii) As $z < 0$, then $-z = 0 + (-z) \in P$. So $-y \cdot z + x \cdot z = (y - x) \cdot (-z) \in P$, hence $x \cdot z > y \cdot z$.

I-5 (i) $1 > 0$ and $-1 < 0$.
Proof I. As $\mathbb{F}$ has more than 1 element, so $1 \neq 0$. Then $1 \in P$ or $1 \in -P$ ($\iff -1 \in P$). It follows from $1 = 1 \cdot 1 = (-1) \cdot (-1)$ that $1 \in P$.
Proof II. Since $1 \neq 0$, we have only one the following holds: $1 \in P$ and $1 \in -P$. If $1 \in -P$, then $-1 \in P$. And if follows that $1 = (-1) \cdot (-1) \in P$ which contradicts to our assumption $1 \in -P$.

I-6 If $x > 0$, then $1/x > 0$.
Proof. Assume contrary, then $1/x \leq 0$. If $1/x = 0$, then $1 = 0 \cdot x = 0$, which is impossible, and hence $1/x < 0$. Then it follows from $x > 0$, we have $1 = x \cdot (1/x) < x \cdot 0 = 0$, which contradicts to I-5.

I-7 If $0 < x < y$, then $0 < 1/y < 1/x$.
Proof. From the given conditions $0 < x < y$, we know that $\dfrac{1}{x} > 0$, $\dfrac{1}{y} > 0$ and $y - x < 0$. So we have $\dfrac{1}{yx} = \dfrac{1}{y} \cdot \dfrac{1}{x} > 0$ and $(y - x) \cdot \dfrac{1}{yx} < 0 \cdot \dfrac{1}{yx} = 0$.
So $\dfrac{1}{x} - \dfrac{1}{y} = \dfrac{y - x}{xy} = (y - x) \cdot \dfrac{1}{xy} < 0$.

I-8 If $z > 0$ and $x \cdot z < y \cdot z$, then $x < y$.
Proof. $x = (x \cdot z) \cdot \dfrac{1}{z} < (y \cdot z) \cdot \dfrac{1}{z} = y$.

Notations. In the following, suppose that a, b, x and y are elements of an ordered field $\mathbb{F}$, and $a < b$. We define notions similar to those of intervals in the field $\mathbb{R}$ of real numbers.

1. Define $U_a = \{ y \in \mathbb{F} \mid a < y \}$ and $L_b = \{ x \in \mathbb{F} \mid x < b \}$. likewise, $(a, b) = U_a \cap L_b$, $[a, b) = \{a\} \cup (a, b)$, and $(a, b] = (a, b) \cup \{b\}$.
$x \in (a, b) \iff (x \in L_b)$ and $(x \in U_a) \iff (a < x < b)$.

2. Suppose that $\mathbb{F}$ is the real number field $\mathbb{R}$ and $a < b$. Denote the *open interval* by $(a, b) = \{ x \in \mathbb{R} \mid a < x < b \}$ and the closed interval $[a, b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}$. There are some other intervals which may be useful too, such as: $(a, b] = \{ x \in \mathbb{R} \mid a < x \leq b \}$ and $[a, b) = \{ x \in \mathbb{R} \mid a \leq x < b \}$. Moreover, one can also relax the finiteness of a or b, then we have the unbounded intervals as follows: $(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b \}$ and $(a, +\infty) = \{x \in \mathbb{R} \mid a \leq x \}$.

3. Though open and closed intervals are simple objects as sets, but they play vital role in the real analysis, as soon as we discuss the concept of limit.

3.5 Absolute Value

1. Let $\mathbb{F}$ be an ordered field, define *absolute value* map $|\cdot| : \mathbb{F} \to \mathbb{F}$ as follows:

$$|a| = \begin{cases} a & \text{if } a > 0 \ \text{ or } a = 0; \\ -a & \text{if } a < 0 \end{cases}.$$

2. If $a, b \in \mathbb{F}$ and $b \geq 0$, then $|a| \leq b$ if and only if $-b \leq a \leq b$.

3. For all x and y in an ordered field $\mathbb{F}$, we have

   (a) $|x| \geq 0$, and that $|x| = 0 \iff x = 0$.

   (b) $|x| = |-x|$;

   (c) $-|x| \leq x \leq |x|$.

   (d) $|xy| = |x| \cdot |y|$;

   (e) $|x + y| \leq |x| + |y|$;

   (f) $|\ |x| - |y|\ | \leq |x - y|$.

   Proof.

   (a) If $x \geq 0$, then $|x| = x \geq 0$; otherwise, $x < 0$, then $|x| = -x = (-1) \cdot x > (-1) \cdot 0 = 0$. The second part follows from trichotomy.

   (b) If x is positive, then $|x| = x$, then $|x| \geq x$ holds.
   If x is negative, i.e. $x < 0$, then after multiplying by (-1) we have $|x| = -x = (-1) \cdot x > (-1) \cdot 0 = 0 > x$. In particular, $|x| \geq x$.
   And the second inequality follows by replacing x by $-x$.

   (c) Divide into 3 cases according x is positive, zero and negative. If x is positive, then $-x$ is negative, so $|-x| = -(-x) = x = |x|$. If x is negative, then $-x$ is positive, then $|-x| = -x = |x|$. If $x = 0$, then $-x = 0$, and so $|x| = 0 = |-x|$.

   (d) If one of x and y is zero, then it is obvious. With loss of generality (WLOG), we may assume that both x and y are not zero. According to trichotomy, We have 3 cases:

      (i) Both x and y are positive, then $xy > 0$, $|x| = x$, $|y| = y$. In this case $|xy| = xy = |x| \cdot |y|$.

      (ii) Both of them are negative, then both $-x$ and $-y$ are positive, and $x \cdot y = (-x) \cdot (-y)$. Hence $|x \cdot y| = |(-x) \cdot (-y)| = |-x| \cdot |-y| = |x| \cdot |y|$.

      (iii) If x and y have different signs, then it follows from $x \cdot y = -(\ (-x) \cdot y\ ) = -(\ x \cdot (-y)\ )$ that $x \cdot y < 0$, so $|x \cdot y| = -(x \cdot y)$. Moreover, because only one of $|x|$ and $|y|$ has a sign different from that x and y, so $|x| \cdot |y| = -(x \cdot y)$. And the result follows.

   (e) Consider the sign of sum $x + y$:
   (i) If $x + y > 0$ then $|x + y| = x + y \leq |x| + |y|$;
   (ii) If $x + y < 0$ then $|x + y| = -(x + y) = (-x) + (-y) \leq |x| + |y|$;
   (iii) If $x + y = 0$ then $|x + y| = |0| = 0 = 0 + 0 \leq |x| + |y|$.
   Remark. Determine when the equality holds.

   (f) $$|\ |x| - |y|\ | \leq |x - y| \iff -(|x - y|) \leq |x| - |y| \leq |x - y|$$
   $$\iff (\ |y| \leq |x| + |x - y|\ ) \wedge (\ |x| \leq |x - y| + |y|\ ).$$
   The first inequality follows from (e) that $|y| = |-y| = |(-x) + (x - y)| \leq |-x| + |x - y| = |x| + |x - y|$, and the second inequality follows from (e) that $|x| = |y + (x - y)| \leq |y| + |x - y|$.