3.6 Dedekind Completeness Axiom and Supremum Principle

1. **Dedekind Completeness Axiom**. An ordered field $\mathbb{F}$ is said to satisfy Dedekind Completness Axiom (DAC), if for any non-empty partition $\{A, B\}$ of $\mathbb{F}$ such that $a < b$ for any $a \in A,\ b \in B$, then there exists exactly one $s \in \mathbb{F}$ such that

   (i) If $u \in \mathbb{F}$ and $u < s$ then $u \in A$, and

   (ii) if $v \in \mathbb{F}$ and $s < v$ then $v \in B$.

2. **Definition**. Let S be a non-empty subset of an ordered field $\mathbb{F}$,
   (i) a number b is called an *upper bound* of S if $x \le b$ for all $x \in S$, and
   (ii) a number c is called an *lower bound* of S if $c \le x$ for all $x \in S$.

3. **Definition**. If $S \subset \mathbb{F}$ has an upper bound (a lower bound) in $\mathbb{F}$ respectively, then S is called *bounded above* (*bounded below*) respectively. If S is called *bounded*, if S has both an upper bound and a lower bound.

   Remark. A subset S of $\mathbb{F}$ is not bounded above, if for any element $a \in \mathbb{F}$, there exists some $x \in S$ such that $x \le s$.

4. **Definition**. Let S be a subset of an ordered field $\mathbb{F}$. An element $s \in \mathbb{F}$ is called the *supremum* or *least upper bound* of S if these two hold:

   (i) s is an upper bound of S;

   (ii) if b is an upper bound of S, then $s \le b$.

5. **Example**. Let $A = \{x \in \mathbb{Q} \mid x \ge 0, \text{ and } x^2 < 2\}$. Prove that (i) A is bounded subset of $\mathbb{R}$; (ii) $\sup A = \sqrt{2}$.
   Proof. (i) For any $x \in A$, we have $x+2 \ge 0+2 > 0$, and $(x-2)(x+2) = x^2 - 4 < 2 - 4 < 0$. It follows from that $x - 2 < 0$ i.e. $x < 2$ for all $x \in A$, and the set A is bounded in $\mathbb{R}$.
   (ii) For any $x \in A$, we have $x^2 < (\sqrt{2})^2$, and $-\sqrt{2} < x < \sqrt{2}$. So $\sqrt{2}$ is an upper bound of A. Let t be any upper bound of A. Want to prove that $t \ge \sqrt{2}$. Assume contrary, that $t < \sqrt{2}$. As $1 \in A$, so $t > 1 > 0$. In this case, $t^2 < (\sqrt{2})^2 = 2$. Define $\rho = \min\{\frac{2-t^2}{2t+1}, 1\}$, so $\rho > 0$ and let $a = t + \rho$. It follows from $0 < \rho < 1$ that $-\rho > -1$ and $-\rho^2 > -\rho$. Consequently, $2 - a^2 = 2 - (t+\rho)^2 = 2 - t^2 - 2t\rho - \rho^2 = 2 - t^2 - 2t\rho - \rho$
   $> (2 - t^2) - \rho(2t+1) \ge (2 - t^2) - \frac{2-t^2}{2t+1}(2t+1) = 0.$
   In particular, $a^2 < 2$, and $a \in A$. Since $\rho > 0$ we have $a > t$, violating that t is an upper bound of A. Hence $\sup A = \sqrt{2}$.

6. **Example**. Let A and B be two non-empty, bounded subset of $\mathbb{F}$. Define $A + B = \{\ a + b \in \mathbb{F} \mid a \in A, b \in B\ \}$. Prove that
   (i) $\sup A \le \sup B$ if $A \subset B$, (ii) $A + B$ is bounded, and
   (iii) $\sup(A + B) \le \sup A + \sup B$.
   Proof. (i) For any $x \in A$, we know that $x \in B$, and hence $x \le \sup B$, i.e. $\sup B$ is also an upper bound of A. By (ii) in the definition of sup, $\sup A \le \sup B$.
   (ii) Let u, l be any upper and lower bound of A respectively, i.e. $l \le a \le u$ for all $a \in A$. Similarly, define u', l' for B, then $l' \le b \le u'$ for all $b \in B$. In particular, $l + l' \le l + b \le a + b \le u + b \le u + u'$, for all $a \in A$ and $b \in B$. So $l + l'$ and $u + u'$ are lower and upper bounds of $A + B$ respectively. Thus $A + B$ is bounded.
   (iii) As in the proof of (ii), one may replace u and u' by $\sup A$ and $\sup B$, so one knows that $\sup A + \sup B$ is an upper bound of the set $A+B$, then it follows from the least upper bound, we have $\sup(A+B) \le \sup A + \sup B$.

7. **Definition**. An element $t \in \mathbb{F}$ is called the *infimum* or *greatest upper bound* of S if the following two conditions hold:

   (i) t is a lower bound of S;

   (ii) if c is a lower bound of S, then $t \ge c$.

8. Definition. Let $S \subset \mathbb{F}$ be nonempty set, $m \in \mathbb{F}$ is called the *maximum* of S, denoted by $\max S$, if m is an upper bound of S, and $m \in S$.