Contents

1 Introduction

1.1 Syllabus

In this course EDUC 250, we are interested in the following topics:

1. Axioms of real number system;
2. Countable and uncountable sets;
3. Limits and continuity of functions of a variable;
4. Axiom of continuity;
5. Nested Interval Theorem;
6. Bolzano-Weirstrass Theorem;
7. Cauchy criterion of convergence and Heine-Borel Thorem;
8. Properties of continuous functions;
9. Elementary theory of differentiation.

1.2 Reference

1. Textbook: W.Rudin, *Principles of Mathematical Analysis*
   有中译本：《数学分析原理》, 机械工业出版社
2. R. Bartle, *Introduction to Real Analysis*.
3. 裘兆泰等编《数学分析学习指导》, 科学出版社
4. 李心讪编《微积分得创立者及其先驱》高等教育出版社
5. 朱匀华等编《数学分析的思想方法》中山大学出版社

2 Elementary Set Theory

1. Let X and Y be two sets. Let $X \times Y = \{ (x, y) \mid x \in X \text{ and } y \in Y \}$ be the set of all pairs of ordered pairs (x, y).

2. Let X and Y be two sets. X and Y are said to have the *same cardinality* or same power if there exists a bijection from X to Y.

3. A subset X is said to be *countable* if X is finite or it has the same cardinality with $\mathbb{N}$. As an example, the set $\mathbb{Z}$ of all integer is countable.

4. Let n be any natural number, and X be a countable set, then X^n is countable.

3 Axioms of Real Number Field

3.1 Binary operations

Let $\mathbb{F}$ be a non-empty set, with two binary operations:

Addition $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}, \quad (x, y) \mapsto x + y$

Multiplication $\times : \mathbb{F} \times \mathbb{F} \to \mathbb{F}, \quad (x, y) \mapsto x \cdot y$

which satisfy the following axioms:

A-1 **Commutative law for addition**.
$x + y = y + x$ for all x and $y \in \mathbb{F}$.

A-2 **Associative law for addition**.
$(x + y) + z = x + (y + z)$ for all x, y and $z \in \mathbb{F}$.
It follows from this axiom that we can write $x + y + z$

A-3 **Existence of zero element**.
There exists a unique element $0 \in \mathbb{F}$, called *zero*, such that $x + 0 = x$ for all $x \in \mathbb{F}$.

A-4 **Existence of additive negative element**.
For each element $x \in \mathbb{F}$, there exists a unique element, denoted by $-x$, such that $x + (-x) = 0$.
Remark. In this moment you shouldn't think $-x$ is given by $(-1) \cdot x$. This symbol has nothing to do with multiplication.

M-1 **Commutative law for multiplication**.
$x \cdot y = y \cdot x$ for all x and $y \in \mathbb{F}$.

M-2 **Associative law for multiplication**.
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all x, y and $z \in \mathbb{F}$.

M-3 **Existence of unity element**.
There exists a unique element $1 \in \mathbb{F}$, called the unity element, such that $x \cdot 1 = x$ for all $x \in \mathbb{F}$.

M-4 **Existence of reciprocals**.
For each $x \in \mathbb{R}$ with $x \neq 0$, there exists a unique element $a^{-1} \in \mathbb{F}$, called the reciprocal of x, such that $x \cdot x^{-1} = 1$.

D **Distributive law**. $x \cdot (y + z) = x \cdot y + x \cdot z$ for any elements $x,\ y,\ z \in \mathbb{R}$,

3.2 Group, Ring and Field

1. The binary operation $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ satisfying Axioms A1-A4 provides an algebraic structure for the set $\mathbb{F}$, called a *group*.

2. The binary operations $+, \ \cdot \ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ satisfying Axioms A1-A4, M1-M2 and D provides an algebraic structure, called a *commutative ring*.

3. A commutative ring satisfying M3-M4 is called a *field*. The set $\mathbb{R}$ of all real numbers $\mathbb{R}$ with usual addition and multiplication is a field. However, $\mathbb{R}$ has more properties than stated above as we will see later.

4. **Theorem**. Let $\mathbb{F}$ be a field, and a, b be two elements in $\mathbb{F}$, then there exists a unique element $x \in \mathbb{F}$ such that $a+x=b$. This element x is given by $x = b+(-a)$, usually denoted by $b-a$.
Proof. Set $x = b+(-a)$, then $a+x = x+a = (\ b+(-a)\ )+a = b+(\ (-a)+a) = b+0 = b$. It remains to prove that the equation $x+a=b$ has a unique solution. Suppose that c and c' are two solutions of the equation $x+a=b$, then $c = c+0 = c+(a+(-a)) = (c+a)+(-a) = b+(-a) = (c'+a)+(-a) = c'+(a+(-a)) = c'+0 = c'$.
Exercise: fill in the reasons why the equalities hold.

5. **Theorem**. Let $\mathbb{F}$ be a field, and a, b be two elements in $\mathbb{F}$ with $a \neq 0$, then there exists a unique element $x \in \mathbb{F}$ such that $a \cdot x = b$. This element x is given by $x = b \cdot (a^{-1})$, usually denoted by b/a.
Proof. Set $x = b \cdot (a^{-1})$, then $a \cdot x = x \cdot a = (b \cdot (a^{-1})\ ) \cdot a = b \cdot (a^{-1} \cdot a) = b \cdot 1 = b$. It remains to prove that the equation $a \cdot x = b$ has a unique solution in x. Suppose that c and c' are two solutions of the equation $a \cdot x = b$, then $c = c \cdot 1 = c \cdot (a \cdot (a^{-1})) = (c \cdot a) \cdot (a^{-1}) = b \cdot (a^{-1}) = (c' \cdot a) \cdot (a^{-1}) = (c' \cdot a) \cdot (a^{-1}) = c' \cdot 1 = c'$.

6. **Theorem**. (**Basic Properties of zero**). Let $\mathbb{F}$ be a field, a and b are two elements in $\mathbb{F}$. Then the following statements hold:

   (1) $a \cdot 0 = 0$. (2) $a \cdot b = 0$ if and only if $a = 0$ or $b = 0$.

   (3) $a \neq 0$ and $b \neq 0$ if and only if $a \cdot b \neq 0$.

   Proof. (1) $a \cdot 0 + a \cdot 0 = a \cdot (0+0) = a \cdot 0$, then $a \cdot 0 = a \cdot 0 + 0 = a \cdot 0 + (\ a \cdot 0 + (-(a \cdot 0)\ ) = (\ a \cdot 0 + a \cdot 0\ ) + (\ -(a \cdot 0)\ ) = (a \cdot 0) + (\ -(a \cdot 0)\ ) = 0$.
(2) If any of a and b is zero, then it follows from (1) that $a \cdot b = 0$. In the following, assume that a and b are both non-zero. Suppose that $a \cdot b = 0$, then by M-4, there exists $b^{-1} \in \mathbb{F}$ such that $b \cdot b^{-1} = 1$. So $a = a \cdot 1 = a \cdot (b \cdot b^{-1}) = (a \cdot b) \cdot b^{-1} = 0 \cdot b^{-1} = 0$, this is impossible.
(3) This is just restatement of (2).

7. **Theorem**. (Basic properties of unity element) Let $\mathbb{F}$ be a field, then the following statements hold:

   (1) If there are at least 2 elements in $\mathbb{F}$, then $1 \neq 0$;

   (2) $(-1) \cdot a = -a$ for all $a \in \mathbb{F}$;

   (3) $a/1 = a$ for all $a \in \mathbb{F}$.

   Proof. (1) Suppose contrary, i.e. $1 = 0$, then $a = a \cdot 1 = a \cdot 0 = 0$ for all $a \in \mathbb{F}$. So $\mathbb{F} = \{0\}$, and this is a contradiction.
(2) $(-1) \cdot a + a = (-1) \cdot a + 1 \cdot a = (-1+1) \cdot a = 0 \cdot a = 0$, and hence, $-a = (-1) \cdot a$.
(3) Let $x = a/1$, then it follows from the theorem that $x = 1 \cdot a = a$. So we have $a/1 = a$ for all $a \in \mathbb{F}$.

8. **Remark**. Though real number field $\mathbb{R}$ is the one we are most familiar with, but here is an exotic example of field: $\mathbb{F} = \{0, 1\}$ in which addition is defined as $0+0=0,\ 1+0=1$ and $1+1=0$; and multiplication is defined as $0 \cdot 0 = 0, 1 \cdot 1 = 1$ and $1 \cdot 0 = 0$. This example serves a very important in algebra and coding theory.

3.3 Ordered Field

1. **Definition**. A field $\mathbb{F}$ is called an *ordered field*, if there exists an non-empty subset P satisfying the following two conditions:

   C-1 For any $x \in \mathbb{F}$, exactly one of the following three alternatives holds: (i) $a \in P$; (ii) $a = 0$; (iii) $-a \in P$.
   This is equivalent to say that the subsets P, $\{0\}$, and $-P = \{\ -x \mid x \in P\ \}$ form a partition of the set $\mathbb{F}$. Sometimes, this is axiom of trichotomy.

   C-2 If $x,\ y \in P$, then $x+y \in P$ and $x \cdot y \in P$.
   This means that the set P is closed under addition and multiplication.

2. **Definition**. Given $a, b \in \mathbb{F}$, define $a < b$ if $b - a \in P$. An element $x \in \mathbb{F}$ is called *positive* (*negative*), if $x \in P$ ($x \in -P$) respectively.

3.4 Properties of Inequalities

For all $x, y, z \in \mathbb{F}$, we have

I-1 Transitivity: If $x < y$ and $y < z$, then $x < z$.
$(x < y) \wedge (y < z) \Longleftrightarrow (y - x \in P) \wedge (z - y \in P) \implies z - x = (z - y) + (y - x) \in P \iff x < z$.

I-2 Trichotomy: Exactly one of the following relations $x < y$, $x = y$, $x > y$ holds.

I-3 If $x < y$, then $x + z < y + z$.
And we have $x + z < y + z \implies x = (x + z) + (-z) < (y + z) + (-z) = y$.
It follows that $x + z < y + z$ is equivalent $x < y$.

I-4 (i) If $x < y$ and $z > 0$ then $x \cdot z < y \cdot z$.
(ii) If $x < y$ and $z < 0$ then $x \cdot z > y \cdot z$.

I-5 (i) $1 > 0$ and $-1 < 0$.
Since $1 \neq 0$, we have only one the following holds: $1 \in P$ and $1 \in -P$. If $1 \in -P$, then $-1 \in P$. And if follows that $1 = (-1) \cdot (-1) \in P$ which contradicts to our assumption $1 \in -P$.

I-6 If $x > 0$, then $1/x > 0$.
Assume contrary, then $1/x < 0$ or $1/x = 0$. It follows from $x > 0$ that $x \neq 0$, i.e. $x \in \mathbb{F} \setminus \{0\}$, so $1/x \in \mathbb{F} \setminus \{0\}$ and $x \neq 0$. Then $1/x < 0$ holds, it follows from I-4(i) $1 = (1/x) \cdot x < 0 \cdot x = 0$ which contradicts to I-5.

I-7 If $0 < x < y$, then $0 < 1/y < 1/x$.
From the given conditions $0 < x < y$, we know that $\dfrac{1}{x} > 0$, $\dfrac{1}{y} > 0$ and $y - x < 0$. So we have $\dfrac{1}{yx} = \dfrac{1}{y} \cdot \dfrac{1}{x} > 0$ and $(y - x) \cdot \dfrac{1}{yx} < 0 \cdot \dfrac{1}{yx} = 0$. So $\dfrac{1}{x} - \dfrac{1}{y} = \dfrac{y - x}{xy} = (y - x) \cdot \dfrac{1}{xy} < 0$.

I-8 If $z > 0$ and $x \cdot z < y \cdot z$, then $x < y$.
$x = (x \cdot z) \cdot \dfrac{1}{z} < (y \cdot z) \cdot \dfrac{1}{z} = y$.

Notations. In the following, suppose that a, b, x and y are elements of an ordered field $\mathbb{F}$.

1. We also write $x > y$ to represent $y < x$, and we write $x \leq y \Longleftrightarrow (x = y)$ or $(x < y)$. Similarly, $x \geq y \Longleftrightarrow (x = y)$ or $(y < x)$. One can easily write down the analogy inequalities with $\leq$ instead of $<$.

2. In the following, we will assume that the ordered field is the field $\mathbb{R}$ of real numbers. As usual, if $a < b$, then we denote the *open interval* by $(a, b) = \{ x \in \mathbb{R} \mid a < x < b \}$, which is the set of real numbers lying between a and b, but not the end points. Similarly, we have the closed interval $[a, b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}$, which is the set of real numbers lying between a and b, and the end points a and b.

3. There are some other intervals which may be useful too, such as: $(a, b] = \{ x \in \mathbb{R} \mid a < x \leq b \}$ and $[a, b) = \{ x \in \mathbb{R} \mid a \leq x < b \}$. Moreover, one can also relax the finiteness of a or b, then we have the unbounded intervals as follows: $(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b \}$ and $(a, +\infty) = \{x \in \mathbb{R} \mid a \leq x \}$.

4. Though open and closed intervals are simple objects as sets, but they play vital role in the real analysis, as soon as we discuss the concept of limit.

3.5 Absolute Value

1. Let $\mathbb{F}$ be an ordered field, define *absolute value* map $|\cdot| : \mathbb{F} \to \mathbb{F}$ as follows:

$$a = \begin{cases} a & \text{if } a > 0 \ \text{ or } a = 0; \\ -a & \text{if } a < 0 \end{cases}.$$

2. If $a, b \in \mathbb{F}$ and $b \geq 0$, then $|a| \leq b$ if and only if $-b \leq a \leq b$.

3. For all x and y in an ordered field, we have

   (a) $|xy| = |x| \cdot |y|$;

   (b) $|x + y| \leq |x| + |y|$;

   (c) $|\,|x| - |y|\,| \leq |x - y|$.

3.6 Dedekind Completeness Axiom and Supremum Principle

1. **Dedekind Completeness Axiom**. An ordered field $\mathbb{F}$ is said to satisfy Dedekind Completness Axiom (DAC), if for any non-empty partition $\{A, B\}$ of $\mathbb{F}$ such that $a < b$ for any $a \in A,\ b \in B$, then there exists exactly one $s \in \mathbb{F}$ such that

   (i) If $u \in \mathbb{F}$ and $u < s$ then $u \in A$, and

   (ii) if $v \in \mathbb{F}$ and $s < v$ then $v \in B$.

2. **Definition**. Let S be a non-empty subset of $\mathbb{F}$,
   (i) a number b is called an *upper bound* of S if $x \leq b$ for all $x \in S$, and
   (ii) a number c is called an *lower bound* of S if $c \leq x$ for all $x \in S$.

3. **Definition**. If $S \subset \mathbb{F}$ has an upper bound (a lower bound) in $\mathbb{F}$ respectively, then S is called *bounded above* (*bounded below*) respectively. If S is called *bounded*, if S has both an upper bound and a lower bound.

   Remark. A subset S of $\mathbb{F}$ is not bounded above, if for any element $a \in \mathbb{F}$, there exists some $x \in S$ such that $x \leq s$.

4. **Definition**. Let S be a subset of an ordered field $\mathbb{F}$. An element $s \in \mathbb{F}$ is called a *supremum* or *least upper bound* of S if the following two conditions hold:

   (i) s is an upper bound of S;

   (ii) if b is an upper bound of S, then $s \leq b$.

5. **Definition**. An element $t \in \mathbb{F}$ (if exists) is called a *infimum* (or *greatest upper bound* of S if the following two conditions hold:

   (i) t is a lower bound of S;

   (ii) if c is a lower bound of S, then $s \geq c$.

6. **Definition**. An ordered field $\mathbb{F}$ is said to satisfy the *supremum principle* if any bounded above nonempty subset S of $\mathbb{F}$ has $\sup S \in \mathbb{F}$.

7. **Theorem**. For any ordered field $\mathbb{F}$, supremum principle follows from Dedekind completeness axiom.

   Proof. Let B be the set of all upper bounds of S, then $B \neq \emptyset$. Let $A = \mathbb{F} \setminus B$. Since $S \neq \emptyset$, choose an element $x \in S$. Then $x - 1 < x$, and so $x - 1 \notin B$, i.e. $x - 1 \in A$ and $A \neq \emptyset$.

   Obviously $A \cup B = \mathbb{F}$ and $A \cap B = \emptyset$. Suppose that $a \in A$ and $b \in B$, then by definition, a is not an upper bound of the set S, so there exists an element $x \in S$ so that $a \leq x$. Moreover, b is an upper bound of S, and so $x \leq b$, it follows that $a \leq x \leq b$. It follows from $A \cap B = \emptyset$ that $a < b$.

   Then from DCA, there exists an element $s \in \mathbb{F}$ so that s separates the two subset A and B. It remains to prove that $s = \sup S$ as follows:

   (a) Suppose that $x \in S$, want to prove $x \leq s$. Because $s \in \mathbb{F} = A \cup B$, so we $s \in A$ or $s \in B$. If $s \in A$, then s is not an upper bound of S, so there exists $x \in S$ so that $s < x$. But then the real number $a = \dfrac{s + x}{2} > \dfrac{s + s}{2} = s$, and hence by the DCA, we have $a \in B$, i.e. a is an upper bound of S. However, $a = \dfrac{s + x}{2} < \dfrac{x + x}{2} = x$, which is contradiction. So $x \leq s$, i.e. s is an upper bound of S.

   (b) Suppose that c is an upper bound of S, we want to prove $c \geq s$. Assume contrary, that is $c < s$, then it follows from DCA that $c \in A$, which contradicts to the assumption that c is an upper bound of S.

Remark. In fact, the supremum principle is equivalent to Dedekind completeness axiom, as we will prove later.

8. **Theorem**. Suppose that ordered field $\mathbb{F}$ satisfies the supremum principle, then Dedekind completeness axiom also holds in $\mathbb{F}$.

   Proof. Let A and B be two non-empty subsets of $\mathbb{F}$ satisfying the conditions stated in Dedekind completeness axiom. Want to show there exists an element $s \in \mathbb{F}$ satisfying (i) and (ii) in (3.5.1). Since $B \neq \emptyset$, choose any element $b \in B$. Then it follows from the given conditions on A and B, we have $a < b$ for all $a \in A$. In particular A is bounded above. Then by Supremum principle, $s = \sup A$ exists.
   Now we prove that s satisfies (i). For any $u \in \mathbb{F}$ with $u < s$, needs to show that $u \in A$. Suppose contrary, then $u \in B$, so u is an upper bound of A as in above. As s is the supremum (least upper bound), we know that $s \leq u$, which is violates $u < s$. Then we prove that s satisfies (ii). For any $v \in \mathbb{F}$ with $s < v$, needs to show $v \in B$. Suppose contrary, then $v \in A$. But s is an upper bound of S, we know $v \leq s$, which violates the condition on v.

9. **Corollary**. Suppose that S be a non-empty subset of an ordered field $\mathbb{F}$ satisfying Dedekind completeness axiom, and S is bounded below, then S has an infimum in $\mathbb{F}$.

10. **Theorem**. Let S be an non-empty subset of an ordered $\mathbb{F}$, then $s = \sup S$ if and only if the following two conditions hold:

    (i) $x \leq s$ for all $x \in S$, and

    (ii) for each $\varepsilon > 0$, there exists $x \in S$ such that $x > s - \varepsilon$.

    Proof. Assume that (i) and (ii) hold, want to prove $s = \sup S$. First, from (i) we know that s is an upper bound of S. Suppose that b is an upper bound of S, want to prove $b \geq s$. Assume contrary, i.e. $b < s$, then let $\varepsilon = (s - b)/2 > 0$, by (ii) there exists $x \in S$ such that $x > s - \varepsilon = s - (s - b)/2 = (s + b)/2 > (b + b)/2 = b$, which contradicts to the fact b is an upper bound of S. Hence $b \geq s$.
    Assume that $s = \sup S$, want to show that both (i) and (ii) hold.
    As s is an upper bound of S, it suffices to establish (ii). Assume that (ii) fails, then there exists a real number $\varepsilon_0 > 0$, such that for any $x \in S$ we have $x \leq s - \varepsilon_0$. So $s - \varepsilon_0$ is also an upper bound of S, less than s, which contradict to the fact that $s = \sup S$.

11. **Definition**. $\mathbb{R}$ is called the *field of real numbers*, if $\mathbb{R}$ is an ordered field satisfying supremum principle. Any element in $\mathbb{R}$ is called a *real number*.

4 Natural Numbers

We are reversing the set-theoretic construction of the real numbers. Suppose that $\mathbb{R}$ has been constructed, and we want to describe some of its subsets, the set of natural numbers, and the set of integers, for example.

1. **Definition**. A subset I of $\mathbb{R}$ is called an *inductive* set if the following two conditions are satisfied: (i) $1 \in I$, and (ii) if $n \in I$, then $n + 1 \in I$. Let $\mathfrak{I}$ be the family of the all inductive subsets I of $\mathbb{R}$, and $\mathbb{N} = \bigcap_{I \in \mathfrak{I}} I$. The elements of $\mathbb{N}$ are called natural numbers.
   From its definition, $\mathbb{N}$ is the smallest (in the inclusion sense) inductive subset of $\mathbb{R}$.

2. **Theorem. Mathematical Induction Principle**.
   Let S be an non-empty subset of $\mathbb{N}$ such that

   (a) $1 \in S$;

   (b) If $n \in S$, then $n + 1 \in S$.

   Then $S = \mathbb{N}$.
   Proof. By definition, S is an inductive subset of $\mathbb{R}$, hence $\mathbb{N} \subset S$. Moreover, if follows from $S \subset \mathbb{N}$ that $S = \mathbb{N}$.

Remark. In most secondary mathematical textbooks, the number 0 is also regarded as a natural number. In this case, we just need to modify the axiom from $1 \in I$ to $0 \in I$. The rest is basically unchanged when we discuss the inductive subset of $\mathbb{R}$. But one has to pay attention when we discuss the other properties of the set of natural numbers, sometimes one needs to modify the statements or propositions a little bit.

3. **Theorem**. (**Archimedean Order Property**).
If $a, b \in \mathbb{R}$, and $a > 0$, then there exists $n \in \mathbb{N}$ such that $na > b$. In particular, the set $\mathbb{N}$ is not bounded above in $\mathbb{R}$.
Proof. Assume contrary, there exist real numbers a and b, with $a > 0$, such that $na \leq b$ for all natural number $n \in \mathbb{N}$. Let $S = \{ na \mid n \in \mathbb{N} \}$, then $a \in S$ and that b is an upper bound of S. So the supremum of S exists in $\mathbb{R}$, and is denoted by s. Since $a > 0$, then there exists $n \in \mathbb{N}$, such that $na > s - a$, i.e. $(n+1)a > s$. As $(n+1)a \in S$, this contradicts to s is the upper bound of S.
Finally, if we take $a = 1 > 0$, then it follows from the first part that for any $b \in R$, there exist $n \in \mathbb{N}$ so that $n = n \cdot a > b$.

Remark. There exist a non-archimedean complete field, of course, it is not the field of real numbers. But archimedean property is very important in elementary analysis, however, it is also useful to know which property of $\mathbb{R}$ requires archimedean assumption.

4. **Theorem**. For each $n \in \mathbb{N}$, we have:

   (a) $1 \leq n$;

   (b) If $n > 1$, then $n - 1 \in \mathbb{N}$;

   (c) If x is a positive real number, and $x + n \in \mathbb{N}$, then $x \in \mathbb{N}$.

   (d) If m, n are natural numbers such that $m > n$, then $m - n \in \mathbb{N}$.

   (e) If $a \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $n - 1 < a < n$, then $a \notin \mathbb{N}$.

Proof. (a) It follows from $\{ x \in \mathbb{R} \mid x \geq 1 \}$ is an inductive subset of $\mathbb{R}$, and hence $\mathbb{N}$ is a subset of this interval.

(b) Define $S_0 = \{n \in \mathbb{N} \mid n-1 \in \mathbb{N} \}$ and $S = \{1\} \cup S_0$. We want to prove that $S = \mathbb{N}$. Obviously, $S \subset \mathbb{N}$, and $1 \in S$. Suppose that $n \in S$ then either $n = 1$ or $n \in S_0$. In the first case $2 = 1 - 1 \in \mathbb{N}$, and so $2 \in S_0$. And in latter case, we have $n - 1 \in \mathbb{N}$. So if follows from inductive assumption, we have $(n+1) - 1 = n = (n-1) + 1 \in \mathbb{N}$. Then we have $n + 1 \in S_0$, and so $n + 1 \in S$. In any cases, $n + 1 \in S$. Then by the principle of mathematical induction, $S = \mathbb{N}$. So $n - 1 \in \mathbb{N}$ provided n (> 1) is a natural number.

(c) Given any positive real number x, let $T = \{ n \in \mathbb{N} \mid \text{(c) holds} \}$. We want to show $T = \mathbb{N}$. First if $x + 1 \in \mathbb{N}$, then $x + 1 > 0 + 1 = 1$ and by (b) we know that $x = (x + 1) - 1 \in \mathbb{N}$, so we have $1 \in T$. Second, we suppose $n \in T$, want to show that $n + 1 \in T$.
For this we assume that $x+(n+1) \in \mathbb{N}$, then we know that $x+n+1 > 0+n+1 > 1$ and hence by (b) we know that $x+n = x+(n+1)-1 \in \mathbb{N}$. From induction hypothesis we know that $n \in T$, it means that if $x + n \in \mathbb{N}$, then $x \in \mathbb{N}$. So $x \in \mathbb{N}$, and $n + 1 \in T$. In this case $T = \mathbb{N}$.

(d) Let $x = m - n$, then $x > 0$. So $(m - n) + n = m \in \mathbb{N}$, and it follows from (c) that $m - n \in \mathbb{N}$.

(e) Suppose contrary, i.e. $a \in \mathbb{N}$. Then from $n - 1 < a < n$, we have $n < a + 1$ and $0 < (a + 1) - n < (n + 1) - n < 1$, which is impossible for natural integers.

5. **Theorem**. (**Well-Ordering Property of** $\mathbb{N}$).
If $A \subset \mathbb{N}$ is non-empty, then A has a smallest element, i.e. $\inf A \in A$.
Proof. Suppose contrary, i.e. A has no smallest element. Let $S = \{ n \in \mathbb{N} \mid n < a \text{ for some } a \in A \}$. We will prove that $S = \mathbb{N}$.

   (i) First we prove that $1 \in S$; otherwise, $1 \notin S$ i.e. for all $a \in A$ we have

$1 \geq a$. In this case, it follows from (a) in the previous theorem that $A = \{1\}$, and $\inf A = 1$, which violate our original assumption.

(ii) Suppose that $n \in S$, want to prove that $n+1 \in S$. Assume contrary, $n+1 \notin S$. Then for all $a \in A$, we have $n+1 \geq a$. But it follows from $n \in S$, there exist some $b \in A$ so that $n < b$. Consequently, $n < b < n+1$ which violates (e) of the previous theorem. In this case $n+1 \in S$.

Finally, A is non-empty, there exists $a \in A$. Because $A \subset \mathbb{N} = S$, so $a \in S$. In particular, $a < a$, which is a contradiction.

6. **Theorem**. If $m, n \in \mathbb{N}$, then $(m+n)$ and $m \cdot n \in \mathbb{N}$.

   Proof. Fix $m \in \mathbb{N}$ and let $S = \{ n \in \mathbb{N} \mid (m+n) \in \mathbb{N} \}$, and similarly $T = \{ n \in \mathbb{N} \mid m \cdot n \in \mathbb{N} \}$. Obviously S and T are subsets of $\mathbb{N}$. In the following, we prove $S = T = \mathbb{N}$ by means of mathematical induction.

   By means of inductive set and $x \cdot 1 = x$, we know that $1 \in S$ and $1 \in T$. Suppose that $n \in S$, i.e. $(m+n) \in \mathbb{N}$, then inductive set $m + (n+1) = (m+n)+1 \in \mathbb{N}$. Suppose that $n \in T$, i.e. $m \cdot n \in \mathbb{N}$, then by distributive law of multiplication, we have $m \cdot (n+1) = m \cdot n + m \in \mathbb{N}$, while the latter holds because of the addition of natural numbers is closed which has just been established.

7. **Definition**. A real number x is called an *integer* if exactly one of the following holds: $x = 0, x \in \mathbb{N}$ or $-x \in \mathbb{N}$. The set of all integers is denoted by $\mathbb{Z}$.

8. **Theorem**. For every real number x, there exists a unique integer $n \in \mathbb{Z}$ such that $n \leq x \leq n+1$. This integer n is usually denoted by $[x]$, called the integral part of x.

   Proof. Suppose that $x \geq 0.$, Let $S = \{ n \in \mathbb{N} \mid n \leq x\}$. Then $0 \in S$, and hence it is a non-empty subset of $\mathbb{N}$, and is always bounded above by x. By the well-order property of $\mathbb{N}$, we know that $\max S$ exists and is also in S too. let $n = \max S$. Then we known that $n \leq x$, and that $n+1 \notin S$, if follows that $n+1 > x$.

   It remains to prove the uniqueness. If m is another integer satisfying $m \leq x < m+1$. Suppose that $m \neq n$, without loss of generality, we may assume that $n < m$, then because of natural numbers, we have $n+1 \leq m$. It follows from the conditions on m and n, we have $x < n+1 \leq m \leq x$, which implies $x < x$, but this is impossible.

   Return to the other case that $x < 0$, then $-x > 0$, and so we have an integer m satisfying $m \leq -x < m+1$. Then $-m-1 < x \leq -m$ we can take $n = -m-1$ if $x \notin \mathbb{Z}$ and $n = -m$ if $x = -m$.

4.1 Rational and Irrational Numbers

1. **Definition**. A real number x is called a *rational number* if x can be expressed in the form $x = \frac{i}{k}$, where $k \neq 0$ and $l, k \in \mathbb{Z}$. The set of all rational numbers is denoted by $\mathbb{Q}$.

2. **Definition**. A real number x is called an *irrational number* if it is not a rational number, i.e. it can be expressed as a fractions with integral denominator and numerator. So the set of irrational numbers is given by $\mathbb{R} \setminus \mathbb{Q}$.

3. **Theorem**. (**Density of Rational Numbers**) For any real numbers x and $y \in \mathbb{R}$ $(x < y)$, there exists at least a rational number $r \in \mathbb{Q}$ such that $x < r < y$.

   Proof. According to Archimedean principle, there exists a natural number $m \in \mathbb{N}$ such that $0 < \frac{1}{m} < y - x$. Let $n = [mx]+1 \in \mathbb{Z}$, then we have $n - 1 \leq mx < n$, and hence, $x < \dfrac{n}{m} = \dfrac{[mx]+1}{m} \leq \dfrac{mx+1}{m} = x + \dfrac{1}{m} < y$. Let $r = \dfrac{n}{m}$, we have $x < r < y$.

4. **Theorem**. (**Density of Irrational Numbers**) For any real numbers x and $y \in \mathbb{R}$ $(x < y)$, there exists at least an irrational number $r \in \mathbb{R} \setminus \mathbb{Q}$

such that $x < r < y$.

Proof. Apply previous to $x/\sqrt{2}$ and $y/\sqrt{2}$, then there exist $s \in \mathbb{Q}$ such that $x/\sqrt{2} < s < y/\sqrt{2}$. Let $r = s\sqrt{2}$, if follows that $x < r < y$, where r is irrational.

5. **Remark**. One can easily verify that $\mathbb{Q}$ is a field under the usual addition and multiplication. Moreover, it is well-known that $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$, so $\mathbb{Q}$ is a proper subset of $\mathbb{R}$.

4.2 Square root and nth root

1. Show that there exists a positive number $x \in \mathbb{R}$ such that $x^2 = 2$.

Solution. Let $S = \{t \in \mathbb{R} \mid t > 0 \text{ and } t^2 \leq 2\ \}$.

(a) Then $1 \in S$, then S is non-empty.

(b) For any real number $t \in S$, we have $t > 0$ and $t^2 \leq 2 < 4$, so $(t-2)(t+2) < 0$. As we know that $t + 2 > 0 + 2 = 2 > 0$, and hence $t - 2 < 0$, i.e. $t < 2$. Hence 2 is an upper bound of S.

(c) Then $\sup S$ exists, and is denoted by x.

It remains to show that $x^2 = 2$. We know that $1 \in S$ and hence $1 \leq x$.

(i) Suppose that $x^2 < 2$, then $\dfrac{2-x^2}{2x+1} > 0$. It follows from the Archimedean property, there exists positive integer n such that $\dfrac{1}{n} < \dfrac{2-x^2}{2x+1}$. Thus $\left(x + \dfrac{1}{n}\right)^2 = x^2 + \dfrac{2x}{n} + \dfrac{1}{n^2} \leq x^2 + \dfrac{2x+1}{n} < x^2 + (2 - x^2) = 2$. So $x + \dfrac{1}{n} \in S$, which violates the definition of $x = \sup S$.

(ii) If $x^2 > 2$, then $x^2 - 2 > 0$ and hence $x > \dfrac{x^2-2}{2x} > 0$. It follows from the definition of $x = \sup S$ there exists an element $y \in S$ such that $y > x - \dfrac{x^2-2}{2x} > 0$. Thus we have $y^2 > \left(x - \dfrac{x^2-2}{2x}\right)^2 = x^2 - (x^2 - 2) + \dfrac{(x^2-2)^2}{4x^2} > 2$, this violates $y \in S$.

Remark. For any positive number $a > 0$, one can follow the same idea to prove that there exists a unique positive number b so that $b^2 = a$ The number b is called the *positive square root* of a.

2. **Example**. Show $\sqrt{2}$ is not a rational number.

Solution. We need to use the unique factorization property of positive integer which we are not going to discuss in details. Suppose contrary, then $\sqrt{2} = \frac{m}{n}$, where m and n are relative prime integers and $n > 0$. Rewrite the equality in the set of integers, instead of rational numbers, so we have $2n^2 = (\sqrt{2}n)^2 = m^2$. Now we have $2 \mid 2n^2 = m^2$, and hence $2 \mid m^2$. Then 2 is a prime, and hence $2 \mid m$. So we rewrite $m = 2k$, where $k \in \mathbb{Z}$. Then the equation becomes $2n^2 = (2k)^2 = 4k^2$, i.e. $n^2 = 2k^2$. Repeat the same argument as before, and so $2 \mid n$. In particular, $2 \mid \gcd(m, n) = 1$, which is impossible.

Remarks. (i) When we assert that $2 \mid m^2$ implies $2 \mid m$, UF (Unique Factorization) property of $\mathbb{Z}$ is essential.

(ii) One can prove that UF property by means by induction.

3. **Theorem**. For any positive real number x, and every integer $n > 0$ there exists one and only one positive y such that $y^n = x$.

Proof. (Uniqueness). If y_1 and y_2 are two positive numbers such that $y_1^n = x = y_2^n$, then $0 = y_1^n - y_2^n = (\ y_1 - y_2\ )(\ y_1^{n-1} + y_1^{n-2}y_2 + \cdots + y_2 y_2^{n-2} + y_2^{n-1}\ )$. As the second factor is positive, then we have $y_1 - y_2 = 0$, and hence $y_1 = y_2$.

(Existence). Let $E = \{\ s \in \mathbb{R} \mid s > 0 \text{ and } s^n < x\}$. Want to prove that $\sup E$ is the n-th root of x. Let $t = \frac{x}{x+1}$. It is obviously that $t < x$ and that $0 < t < 1$. Hence we have $t^n < t < x$. In particular, $E \neq \emptyset$.

Now we want to prove that $1 + x$ is an upper bound of E. For any real number $a > 1 + x$, want to prove that then $a > 1 + x > 1$ and hence $a^n > a > 1 + x > x$, so that $a \notin E$. In particular, we have $s^n \leq x < a^n$, so $s < a$. So $1 + x \geq s$, i.e. $1 + x$ is an upper bound of E.

By supremum principle and $t > 0$, we know that $y = \sup E$ exists, and is a positive number. It remains to show that $y^n = x$. Suppose contrary, then we have two cases: (i) $y^n < x$; and (ii) $y^n > x$.

(i) If $y^n < x$, then choose $h \in (0,1)$ such that $h < \dfrac{x-y^n}{n(y+1)^{n-1}}$. Want to prove that $y+h \in E$ as follows:
$(y+h)^n - y^n$
$< [(y+h)-y] \cdot [(y+h)^{n-1} + (y+h)^{n-2}y + \cdots + (y+h)y^{n-2} + y^{n-1}]$
$< h[(y+h)^{n-1} + (y+h)^{n-1} + \cdots + (y+h)^{n-1}] = nh(y+h)^{n-1} < nh(y+1)^{n-1} = x - y^n$. And hence $(y+h)^n < x$, i.e. $y+h \in E$ which violates that $y = \sup E$.

(ii) If $y^n > x$, set $k = \dfrac{y^n - x}{ny^{n-1}}$. Then $0 < k = \dfrac{y^n - x}{ny^{n-1}} < \dfrac{y^n}{ny^{n-1}} = \dfrac{y}{n} \leq y$. If $a \geq y-k$, then $y^n - a^n \leq y^n - (y-k)^n < kny^{n-1} = y^n - x$. Thus $a^n > x$, and $a \notin E$. It follows that $y-k$ is an upper bound of E, which violates that y is the least upper bound of E.

Hence $y^n = x$.

Remark. As the n-th root is an operation on the set $\mathbb{R}$ of real numbers, like the arithmetic operations, it satisfies several algebraic identities. We leave these as your exercises to check these identities.

5 Heine-Borel Theorem

1. **Definition**. A family $\mathcal{F}$ of open intervals is a set of open intervals, i.e. each element $I \in \mathcal{F}$ is an open interval (a,b). Sometimes, we write $\mathcal{F} = \{ U_\alpha \}_{\alpha \in J}$ if we want to indicate the open intervals U_α, where α is the index from the indexed set J.

2. **Definition**. A *subfamily* $\mathcal{F}_0$ of $\mathcal{F}$ means that every member in $\mathcal{F}_0$ is a member of $\mathcal{F}$. Given $\mathcal{F}$ is a family of subsets in $\mathbb{R}$, a subfamily $\mathcal{G}$ of $\mathcal{F}$ is said to be *finite* if $\mathcal{G}$ has only finite members.

3. **Definition**. Let $S \subset \mathbb{R}$. A family $\mathcal{F}$ of open intervals is said to *cover* (or to be an *open covering* of) S if for each each point $x \in S$, there exists an open interval $I \in \mathcal{F}$ such that $x \in I$, that is $S \subset \bigcup_{I \in \mathcal{F}} I$.

4. **Definition**. $S \subset \mathbb{R}$ is called *compact* if for each family $\mathcal{F}$ of open covering of S, there exists a finite subfamily $\mathcal{F}_0$ of $\mathcal{F}$ such that $\mathcal{F}_0$ covers S.

5. **Theorem**. (Heine-Borel). Finite closed interval $[a,b]$ is a compact set, where $a < b$ are finite numbers.
Proof. Suppose that $\mathcal{F}$ is a family of open intervals covering $[a,b]$. Let $S = \{ x \in [a,b] \mid [a,x]$ can be covered by a finite subfamily of $\mathcal{F} \}$.

(a) First $a \in S$, as F covers $[a,b]$, hence there exists an open set U in $\mathcal{F}$ such that $a \in U$. So $S \neq \emptyset$.

(b) S is bounded since $S \subset [a,b]$. By the supremum principle, $s = \sup S$ exists. It remains to show that $s = b$. Suppose contrary, then $s < b$.
From the covering $\mathcal{F}$, there exists an open interval $U_0 \in \mathcal{F}$ such that $s \in U_0$. Since U_0 is open interval, there exists $d > 0$ such that $(s-d, s+d) \subset U$. It follows from the definition of supremum of S, we have $[a, s_0]$ can be covered by a finite subcover of $\mathcal{F}$, where $s - d < s_0 < s$. So $\{ U_0, U_1, \cdots, U_n \}$ is a finite subcover of $[a, s+d/2]$. But it violates that s is an upper bound of S.

6 Sequence and Subsequence

1. **Definition**. A *sequence* of real (complex) numbers is a function $x : \mathbb{N} \to \mathbb{R}$ ($\mathbb{C}$). We usually denote a sequence by $(x_n)_{n \in \mathbb{N}}$, or $\{ x_n \}_{n \in \mathbb{N}}$, or simply (x_n). The k-th term of sequence (x_n) is given by the value of x_k.
Given a sequence $(x_n)_{n \in \mathbb{N}}$, consider a sequence of increasing positive integers: $n_1 < n_2 < n_3 < \cdots$, then the sequence $(x_{n_i})_{i \in \mathbb{N}}$ is called a *subsequence* of (x_n).

Sometimes, we may extend the concept of sequence defined on $\mathbb{N}$ to the one defined on $\mathbb{N}\times\mathbb{N}$, called double indexed sequence, or to the one defined on $\mathbb{Z}$.

2. **Definition**. Let (x_n) be a sequence of real numbers.

   (a) (x_n) is called *(strictly) increasing* if $x_{n+1} > x_n$ for all $n \in \mathbb{N}$; and *non-decreasing* if $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$.

   (b) (x_n) is called *(strictly) decreasing* if $x_{n+1} < x_n$ for all $n \in \mathbb{N}$; and *non-increasing* if $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$.

   (c) (x_n) is called *monotone*, if it is either strictly increasing or strictly decreasing.

3. **Definition**. A sequence (x_n) of real numbers (complex numbers) is said to converge if there exists $a \in \mathbb{R}$ ($a \in \mathbb{C}$) such that for any given $\varepsilon > 0$ there exists a natural number N such that $|x_n - a| < \varepsilon$ for all $n \geq N$.
   Remark. This definition is very important in modern analysis.

6.1 Equivalent definitions of sequential limit

Given a sequence $(x_n)_{n\geq 1}$ of real numbers, and we say that a is the limit of the sequence $(x_n)_{n\geq 1}$ if one of the following holds:

1. ($\varepsilon - N$ **definition**). For any given $\varepsilon > 0$ there exists a natural number N such that $|x_n - a| < \varepsilon$ for all $n \geq N$.

2. (**Open Neighborhood definition**). For any $\varepsilon > 0$, there are only finitely many terms in the sequence $\{x_n\}$ outside the neighborhood $U(a, \varepsilon)$ of the limit a.

3. (**Subsequence form**). Every subsequence $\{ x_{n_k}\}$ of $\{x_n\}$ is convergent.

6.2 Consequences of the existence of limit

1. The limit of a convergent sequence is unique.
   Proof. Let a and b be the limits of a convergent sequence (x_n), want to prove that $a = b$. Assume contrary, Then let $\varepsilon = |a - b|/2 > 0$, there exist n_1 and n_2 such that $|x_n - a| < \varepsilon$ for $n \geq n_1$ and $|x_n - b| < \varepsilon$ for $n \geq n_2$. Take $N = \max\{n_1, n_2\}$, so $|a - b| = |(a - x_N) - (b - x_N)| \leq |a - x_N| + |b - x_N| < \varepsilon + \varepsilon = |a - b|$, which is impossible.

2. Every convergent sequence is bounded.
   Proof. Let $a = \lim\limits_{n\to\infty} x_n$, take $\varepsilon = 1 > 0$, there exist N such that $|x_n - a| < 1$ for all $n \geq N$. Then $|x_n| = |(x_n - a) + a| \leq |x_n - a| + |a| < 1 + |a|$ for all $n \geq N$. Take $M = \max\{ |x_1|, |x_2|, \cdots, |x_{N-1}|, 1 + |a| \}$, then we have $|x_n| \leq M$ for all $n \in \mathbb{N}$.

3. Deleting, inserting, or modifying finitely many terms of a convergent sequence does not alter its limit.

4. A convergent sequence, under rearrangement, converges to same limit.
   Proof. Let $\pi : \mathbb{N} \to \mathbb{N}$ be a bijective map. Let $y_n = x_{\pi(n)}$, then the sequence (y_n) is a rearrangement of (x_n). For any $\varepsilon > 0$, there exists $N > 0$ such that $|x_n - a| < \varepsilon$ for all $n > N$. Let $M = \max\{\pi^{-1}(1), \pi^{-1}(2), \cdots, \pi^{-1}(N)\}$, then for any $n > M$ we have $\pi(n) > N$ so we have $|y_n - a| = |x_{\pi(n)} - a| < \varepsilon$. Then we have $\lim\limits_{n\to\infty} y_n = a$.

6.3 Divergence Criterion

Theorem. Let (x_n) be a sequence of real numbers. Then the followings are equivalent:

1. The sequence (x_n) does not converge to $x \in \mathbb{R}$.

2. There exists an $varepsilon_0 > 0$ such that for any $k \in \mathbb{N}$, there exists $r_k \in \mathbb{N}$ such that $r_k \geq k$ and $|x_{r_k} - x| \geq \varepsilon_0 > 0$.

3. There exists an $\varepsilon_0 > 0$ and a subsequence (x_{n_k}) such that $|x - x_{n_k}| \geq \varepsilon_0 > 0$ for all $n \in \mathbb{N}$.

6.4 Techniques of establishing the sequential limits

In order to prove that $\lim\limits_{n\to\infty} x_n = a$, one needs to prove the following:
$\forall \varepsilon > 0, \exists N > 0$ such that $\forall n > N$, we have $|x_n - a| < \varepsilon$.[1]

1. (Determine the smallest N.) By solving the inequality $|x_n - a| < \varepsilon$, or by equivalent substitution, it is possible to determine all n satisfying the inequality, then one can choose the minimum $N(\varepsilon)$, so that $\forall n > N$, then $|x_n - a| < \varepsilon$.

2. (Relaxing the bound.) Sometimes, it is difficult to solve the inequality $|x_n - a| < \varepsilon$ explicitly, and hence determine the solution set of all n. In this case, one can simplify the inequality $|x_n - a|$, and even obtain an upper bound in the form of a new function $H(n)$, usually simpler than the original one, such that $|x_n - a| \leq H(n)$. Then it remains to solve the inequality $H(n) < \varepsilon$.
   Remark. However, it could happen that the solution set of $H(n) < \varepsilon$ is a proper subset of that of the original inequality $|x_n - a| < \varepsilon$.

3. (Divide and Conquer.) Sometimes, it is impossible to simplify, or to relax the inequality without any assumption on the values of n. So one can consider sufficiently large $n > N_1$, (N_1 is some fixed integer) so that one can bound $|x_n - a|$ by $H(n)$, and even determine the solution set of $n > N(\varepsilon)$ satisfying the inequality constraint $H(n) < \varepsilon$. Then the desired integer can be set to be $N = \max\{N_1, N(\varepsilon)\ \}$.

[1] 典型问题与方法

7 Continuity Axiom of Real Numbers

1. **Supremum Principle**. Every non-empty subset S of the real number field $\mathbb{R}$, bounded above, has a supremum in $\mathbb{R}$.

2. **Monotone Convergence Theorem**. Bounded montone sequence is convergent.

3. **Nested Interval Theorem**. Suppose that $\{\ I_n \mid n = 1, 2, \cdots, \}$ be a family of intervals satisfying:

   (a) $I_n \supset I_{n+1}$ for $n = 1, 2, \cdots$;

   (b) $\lim\limits_{n\to\infty} |I_n| = 0$, where $|\ [a,b]\ | = b - a$, is the length of the interval $[a, b]$.

   Then the exists an unique real number x such that $x \in I_n$ for all n, and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = x$, i.e. $\bigcap\limits_{n\geq 1} I_n = \{x\}$.

4. **Theorem of Finite Subcover**. If the family $\mathcal{F}$ of open intervals cover the closed interval $[a, b]$, then there exists a finite subcover $\mathcal{G}$ of $\mathcal{F}$ covering $[a, b]$.

5. **Theorem of Accumulation Points**. Every bounded infinite subset of $\mathbb{R}$ has at least an accumulation point.

6. **Bolzano-Weierstrass Theorem**. Every bounded sequence has a convergent subsequence.

7. **Cauchy Criteria of Convergence**. $\{a_n\}_{n\geq 1}$ converges if and only if for any given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - a_m| < \varepsilon$ for all $n, m > N$.

In the following, we will prove that these 7 conditions are equivalent. We start from Nested Interval Theorem to prove that finite subcover exists for any open covering of $[a, b]$.

(NIT $\Rightarrow$ TFS). Suppose contrary, assume that interval $I_1 = [a, b]$ *can not* be covered by any finite subcovers of a covering $\mathcal{F}$ of $[a, b]$. Subdivide the interval $[a, b]$ into two closed subinterval of equal lengths. By assumption at least one I_2 of these two closed subinterval can not be covered by a finite subcover of $\mathcal{F}$. One can use mathematical induction to define a sequence of nested intervals $I_1 \supset I_2 \supset \cdots \supset I_n \cdots$ with length $|I_{k+1}| = |I_k|/2$ as follows:
Suppose that $I_1, I_2 \cdots, I_n$ has been defined, divide the interval I_n into two closed subintervals of equal length.
If both of these two subintervals can be covered by two finite subcovers $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{F}$ respectively, then the union $\mathcal{F}_1 \cup \mathcal{F}_2$ is a finite subcover of $\mathcal{F}$. Going backward, one can see that $I_1 = [a, b]$ can also be covered by a finite subcover $\mathcal{G}$ of $\mathcal{F}$, which violates our first assumption. It follows that one of the two subintervals of I_n has no finite subcover of $\mathcal{F}$, call it I_{n+1} and its length $|I_{n+1}| = |I_n|/2$.
By Nested Interval Theorem, we know that $\cap_{n\geq 1} I_n = \{x_0\}$ where $x_0 \in [a, b]$. Then there exists an interval $J_0 \in \mathcal{F}$ containing the point x_0. As J_0 is an open interval of positive length, there exists a natural number N such that $I_n \subset J_0$ for all $n \geq N$. But it violates that I_n can't be covered by any finite subcover of $\mathcal{F}$.
(TFS $\Rightarrow$ TAP). Let S be a bounded infinite subset of $\mathbb{R}$, want to prove that S has at least an accumulated point. Suppose contrary, i.e. any real number is not an accumulated point of S. As S is bounded, we may assume that $S \subset [a, b]$. In particular, any point x of $[a, b]$ is not an accumulated point of S, then there exist $\varepsilon_x > 0$, depending on x, such that the open interval $U(x) = (x - \varepsilon_x, x + \varepsilon_x)$ intersects S at most one common point x, i.e. $U(x) \cap S \subset \{x\}$. Now we define a covering of $[a, b]$ as follows: $\mathcal{F} = \{ \ U(x) = (x - \varepsilon, x + \varepsilon) \mid x \in [a, b] \ \}$, which is obviously an open covering of $[a, b]$. By Theorem of Finite Subcover, we know that there exist a finite subcover $\{ \ U(x_1), \ U(x_2), \ \cdots, \ U(x_n) \ \}$ of $\mathcal{F}$ covering the interval $[a, b]$. In particular, $\bigcup_{i=1}^{n} U(x_i) \supset [a, b] \supset S$. Now S is infinite, then by pigeonhole principle there exists an open interval $U(x_k)$ containing infinitely many points of S. But it violates the choice of $U(x)$ which contains at most a point in S.
(TAP$\Rightarrow$ BWT). Suppose that (x_n) be a bounded sequence of real numbers. There are two cases:
(i) the sequence (x_n) takes on finitely many values. Then there exists a subsequence taking on a constant value, and hence it is convergent.
(ii) the sequence (x_n) takes on infinitely many values. In this case, the set $S = \{ \ x_n \mid n \geq 1 \ \}$ is a bounded infinite subset of $\mathbb{R}$, and hence by TAP, S has at least an accumulated point a. It remains to construct a convergent subsequence $(x_{n_i})_{i\geq 1}$ with limit a as follows: (i) $x_{i_1} = x_1$; (ii) By definition of a, for any $k \geq 1$ there exists a point x_{i_k} which lies in the punctured interval $(\ a - \frac{1}{k}, a + \frac{1}{k} \ ) \setminus \{a, x_{i_1}, x_{i_2}, \cdots, x_{i_{k-1}} \ \}$.
(BWT $\Rightarrow$ CCC). Let $(x_n)_{n\geq 1}$ be a Cauchy sequence, i.e. for any given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ (depending on the choice of ε) such that $|x_n - x_m| \leq \varepsilon$ whenever $n, m \geq N$. Want to establish the limit of the sequence (x_n). First we prove that the sequence is bounded as follows: Take $\varepsilon = 1$ then there exists n_0 such that $|x_{n_0+1} - x_m| < 1$ for all $m \geq n_0$. In particular, we have $|x_m| < 1 + |x_{n_0+1}|$. Take $M = \max\{|x_1|, |x_2|, \cdots, |x_{n_0}|, |x_{n_0+1}| + 1 \ \}$, then we have $|x_n| \leq M$ for all $n \in \mathbb{N}$. Then by BWT, we know that (x_n) has a convergent sequence $(x_{i_n})_{n\in\mathbb{N}}$ with limit a. It remains to prove $\lim_{n\to\infty} x_n = a$.
For any given $\varepsilon > 0$, there exists $k_1 > 0$ such that $|x_n - x_m| < \varepsilon/2$ for all $n, m \geq k_1$. Moreover, there exist $k_2 > 0$ such that $|x_{i_n} - a| < \varepsilon/2$ for all $n \geq k_2$. Moreover, Take N to be the least i_k in the set $\{k_2, k_2 + 1, \cdots\}$ with $k \geq k_1$. So there exists $m \in \mathbb{N}$ such that $i_m \geq k_2$. Then for any $n \geq N$, we have $|x_n - a| \leq |x_n - x_{i_m}| + |x_{i_m} - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.
(CCC$\Rightarrow$SC). Suppose that S is a nonempety subet of $\mathbb{R}$, which is bounded above. Let b_1 be an upper bounded of S. As $S \neq \emptyset$, choose $a_1 \in S$. Then $a_1 \leq b_1$. If $a_1 = b_1$ then $\sup S = b_1$, nothing to be proved. Otherwise, $a_1 < b_1$. Let $I_1 = [a_1, b_1]$, consider the midpoint m_1 of I_1 : if m_1 is an upper bound of S, define $I_2 = [a_1, m_1]$; otherwise $I_2 = [m_1, b_1]$. Similarly, we can define, by

means of mathematical induction, a sequence of closed interval $I_n = [a_n, b_n]$, with $|I_{n+1}| = |I_n|/2$ and $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$, and $a_n \in S$ and b_n is an upper bound of S. In any case, if $a_n = b_n$ for some $n \in \mathbb{N}$, we know that $\sup S = a_n$, so nothing to be proved, hence we may assume that $a_n < b_n$ for all $n \in \mathbb{N}$.

Now we want to show that $(a_n)_{n\geq 1}$ is a Cauchy sequence. For any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|I_{n_0}| = |I_1|/2^{n_0} < \varepsilon$. Then for any $n \geq m \geq n_0$ we have $[a_m, a_n] \subset [a_m, b_m] \subset [a_{n_0}, b_{n_0}]$. It follows that $|a_n - a_m| = |[a_m, a_n]| \leq |\ [a_{n_0}, b_{n_0}]\ | = |I_{n_0}| = |I_1|/2^{n_0} < \varepsilon$. Then by CCC, we know that (a_n) has a convergent subsequence with limit a Because $|a_n - b_n| = |b_1 - a_1|/2^n$, the corresponding subsequence of (b_n) also converges to the same limit a.

Next we need to show a is the supremum of S. (i) First we will prove that a is an upper bound of S. As b_n is an upper bounded for S for all $n \in N$, so for any fixed $x \in S$, we have $x \leq b_n$ for all $n \in \mathbb{N}$. Thus $x \leq \lim_{n\to\infty} b_n = a$. In particular, a is an upper bound of S. (ii) Next we will show a is the least upper bound of S. For any $\varepsilon > 0$ there exist $N \in \mathbb{N}$ such that $a - a_n = |a_n - a| < \varepsilon$. In particular, $a - \varepsilon < a_n$, where $a_n \in S$. Hence $a = \sup S$.

(SP $\Rightarrow$ MCT). Suppose that (x_n) is a bounded montone sequence. WLOG, we may assume that (x_n) is increasing; otherwise consider $(-x_n)$ instead. Let $S = \{\ x_n \mid n \geq 1\ \}$ be the set of values taken by this sequence. S is bounded as so is (x_n). Then by SP we know $s = \sup S$ exists and is finite. Now we want to prove that $s = \mathit{lim}_{n\to\infty} x_n$. For any $\varepsilon > 0$, there exist an element $x_{n_0} \in S$ such that $x_{n_0} > s - \varepsilon$. Then for any $n \geq n_0$ we have $|x_n - s| = s - x_n < s - x_{n_0} < \varepsilon$.

(MCT$\Rightarrow$NIT) Assume that $\{\ I_n \mid n \geq 1\ \}$ be a family of intervals satisfying the conditions (a) and (b) stated in NIT. Then (a_n) is bounded monotone sequence, then by MCT, we know that the sequence a_n converges to a limit a as $n \to \infty$. Then $\lim_{n\to\infty} b_n = \lim_{n\to\infty} [(b_n - a_n) + (a_n)] = \lim_{n\to\infty} (b_n - a_n) + \lim_{n\to\infty} (a_n) = 0 + a = a$. As $a = \sup\{\ a_n \mid n \geq 1\ \}$, so we have $a_n \leq a$ for all $n \in \mathbb{N}$. Similarly, $a \leq b_n$ for all $n \in \mathbb{N}$. Hence $a \in [a_n, b_n]$ for all $n \in \mathbb{N}$.

8 Functions

8.1 Relations

Definition. Let X and Y be two subsets in $\mathbb{R}$, a *relation* R from X to Y is a subset $G(R)$ of $X \times Y$, called the graph of the relation of R. Sometimes, we identify the relation R with its graph $G(R)$ in $X \times Y$.

Let R be a relation from X to Y, then the subset $D(R) = \{\ x \in X \mid (x, y) \in R$ for some $y \in Y\ \}$ of X is called the *domain* of the relation R; and the subset $\mathrm{Ran}(R) = \{\ y \in Y \mid (x, y) \in R$ for some $x \in X\}$ of Y is called the *range* of the relation of R.

Definition. A *function* f from X to Y, denoted by $f : X \to Y$, is a relation from X to Y satisfying the following conditions:

1. If $(x, y),\ (x, z) \in \mathbb{R}$ then $y = z$, or equivalently
2. i.e. any two ordered pairs of R are the same if they both have the same elements in the first entry.

If the relation R really satisfies the condition of a function, then for any $(x, y) \in R$, we call the element x in the first entry of the ordered pair *the input value*, and call the element y in second entry the *output value* of x. We usually denoted by $f(x)$, which want to point out the dependence relation between y and x.

Definition. A function $f : X \to Y$ from X to Y is called

1. *injective* or *one-to-one* if one of the following holds:
   (a) if $f(x_1) = f(x_2)$ then $x_1 = x_2$; or equivalently,
   (b) if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.
2. *surjective* if the range $\mathrm{Ran}(R) = Y$, or equivalently, if for any $y \in Y$, there exists some $x \in X$ such that $f(x) = y$.
3. *bijective*, if it is both injective and surjective.

4. **Definition**. Let $f : X \to Y$ and $g : Y \to Z$ be two functions. For any $x \in X$, we denote $y = f(x)$ and $z = g(y)$. Define a new function $h : X \to Y$ as follows: $h(x) = z$, where $z = g(y) = g(f(x))$. We usually denote h by $g \circ f$.

5. **Theorem**. Let $f : X \to Y$ be a bijective function, then there exists a function $g : Y \to X$ so that $g \circ f = \mathrm{id}_X$, and $f \circ g = \mathrm{id}_Y$.
   Proof. Define $g : Y \to X$ as follows: for any $y \in Y$, because f is surjective, there exist $x \in X$ such that $f(x) = y$. In fact, such a x is unique because f is injective. Hence define $g(y) = x$. In other words, $g(y) = x$ if and only if $f(x) = y$. It remains to show that (i) $g \circ f = \mathrm{id}_X$, and (ii) $f \circ g = \mathrm{id}_Y$.
   (i) For any $x \in X$, we have $g \circ f(x) = g(f(x)) = x$.
   (ii) For any $y \in Y$, we have $f \circ g(y) = f(g(y)) = y$.
   Both last equalities follows from the definition of g.

6. Let $X \subset \mathbb{R}$ and $f, g : X \to \mathbb{R}$ be functions defined on X. Define functions: $f + g, f - g, f \cdot g : X \to \mathbb{R}$ as follows: (i) $(f + g)(x) = f(x) + g(x)$; (ii) $(f - g)(x) = f(x) - g(x)$; (iii) $(f \cdot g)(x) = f(x) \cdot g(x)$ for all $x \in X$; (iv) $(f/g)(x) = f(x)/g(x)$ provided $g(x) \neq 0$ for all $x \in X$.

7. **Definition**. Let $f : X \to Y$ be a function, S be a subset of X. f is called

   (a) *(strictly) increasing* on S if for all $x > y \in S$, we have $f(x) > f(y)$.

   (b) *non-decreasing* on S if for all $x > y \in S$, we have $f(x) \geq f(y)$.

   (c) *(strictly) decreasing* on S if for all $x > y \in S$, we have $f(x) < f(y)$.

   (d) *non-increasing* on S if for all $x > y \in S$, we have $f(x) \leq f(y)$.

   (e) *monotone* on S if any one of the condition holds.

8.2 Limits of Function

1. **Definition**. Let $\delta > 0$, $a \in \mathbb{R}$, we denote the set $\{ x \in \mathbb{R} \mid |x - a| < \delta \}$ by $B(x, \delta)$ or $B_\delta(x)$, called an open ball of radius δ centered at a.

2. **Definition**. Let $A \subset \mathbb{R}$ be a non-empty subset of $\mathbb{R}$. A point $x \in \mathbb{R}$ is called an *accumulation point* or *limit point* of A if for any given $\varepsilon > 0$, the open interval $(x - \varepsilon, x + \varepsilon)$ contains infinitely many points of E. Equivalently $A \cap (x - \varepsilon, x + \varepsilon) \setminus \{x\} \neq \emptyset$. A point $x \in A$ is called an *isolated point* of A if there exists $\delta > 0$ such that $(x - \delta, x + \delta) \cap A = \{x\}$, i.e. the open interval contains only one point x of A.

3. **Definition**. Let $f : A \to \mathbb{R}$ be a real-valued function defined on $A \subset \mathbb{R}$. Let $a \in A$, then we say the *limit of the function* f *at the point* $a \in A$ equals to l, if the following is satisfied:
   For any $\varepsilon > 0$, there exists $\delta > 0$ such that $| f(x) - l | < \varepsilon$ for any $x \in (a - \delta, a + \delta) \cap A \setminus \{a\}$.
   In this case, we write $\lim\limits_{x \to a} f(x) = l$.
   Remark. If a is an isolated point A, then by definition, $(a - \delta, a + \delta) \cap A \setminus \{a\} = \emptyset$ for some $\delta > 0$, and hence the conditions is meaningless.

4. **Properties of Limits**. Let f, $g : A \to \mathbb{R}$ be functions defined on the same domain A. Suppose that the limits of f and g at the point $a \in A$ is l and m. Then

   (a) $\lim\limits_{x \to a} (f(x) + g(x)) = l + m$;
   (b) $\lim\limits_{x \to a} (f(x) - g(x)) = l - m$;
   (c) $\lim\limits_{x \to a} (c \cdot f(x)) = c \cdot l$;
   (d) $\lim\limits_{x \to a} (f(x) \cdot g(x)) = l \cdot m$;
   (e) $\lim\limits_{x \to a} (\frac{f(x)}{g(x)}) = \frac{l}{m}$, provided that $m \neq 0$.

5. **Definition**. Let $f : A \to \mathbb{R}$ be function defined on A, $a \in A$ then f is called to be *continuous at* a, if the limit $\lim\limits_{x \to a} f(x)$ exists, and is equal to $f(a)$. Equivalently, for any given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ for all $(a - \delta, a + \delta) \cap A$.
   $f : A \to \mathbb{R}$ is called *continuous* on A if f is continuous at every point $a \in A$.

6. **Theorem**. If $f : A \to \mathbb{R}$ is continuous at a, there exists $\delta > 0$ such that f is bounded on a neighborhood $U = B(a, \delta)$ of a.
   Remark. The result is just a local property of continuous function. However, f may not be bounded on the set A.

7. **Theorem. (Dini)** Let $f : A \to \mathbb{R}$ be a function, $a \in A$. Then the limit of the function f at a is equal to l if and only if $\lim_{n\to\infty} f(x_n) = f(a)$, for any sequence (x_n) converging to a.

8. **Theorem**. Let $f,\ g : \mathbb{R} \to \mathbb{R}$ be functions, $x_0 \in \mathbb{R}$. If f is continous at x_0 and g is continuous at $f(x_0)$, then $g \circ f$ is a continuous at x_0.

9. **Theorem**. Let $f : (a, b) \to \mathbb{R}$ be a strictly increasing continuous function on (a, b). Let $J = f[I]$ be the image of I under f. Let $g : J \to I$ be the inverse function of f. Prove that g is a continuous on J.

10. **Definition**. Let $f : A \to \mathbb{R}$ be a function. f is called *uniformly continuous* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for any $x, y \in A$ and $|x - y| < \delta$.

11. **Definition**. Let $C \subset \mathbb{R}$, C is called to be a *compact* set if any open cover of C has a finite subcover.

EDUC-205 Mathematical Analysis I

Course Notes Part I

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