Contents

1 Preliminary Exercises

1.1 Famous Inequalities

1. (**Inequality of absolute value**) Let $a,\ b$ be real numbers, then

   (a) $|x| < h \iff x \in (-h, h)$;

   (b) $|\ |a| - |b|\ | \le |a + b| \le |a| + |b|$;

   (c) $|\ |a| - |b|\ | \le |a + b| \le |a| + |b|$.

2. Let $S = \{\ \frac{a_1}{b_1},\ \frac{a_2}{b_2},\ \cdots,\ \frac{a_n}{b_n}\ \}$, where $b_k > 0$ for all $k = 1, 2, \cdots, n$. Then we have $\min S \le \dfrac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \le \max S$.

3. (a) Let $n \ge 2$, and $a_1, a_2, \cdots, a_n$ be positive numbers, then $(1+a_1)(1+a_2)\cdots(1+a_n) > 1 + (a_1 + \cdots + a_n)$.

   (b) Let $n \ge 2$, and $a_1, a_2, \cdots, a_n$ be positive numbers all less than 1 , then $(1-a_1)(1-a_2)\cdots(1-a_n) > 1 - (a_1 + \cdots + a_n)$.

4. If $0 < a < 1$ and n is a natural number, then $1 + a + a^2 + \cdots + a^n < \frac{1}{1-a}$.

5. (**Bernoulli's Inequality**) Suppose that $a > 0$ or $-1 < a < 0$, and $n \ge 2$ is an integer, then $(1+a)^n > 1 + na$.

6. If $n \ge 2$ is an integer, then
   (a) $\left(1 + \frac{1}{n-1}\right)^{n-1} < \left(1 + \frac{1}{n}\right)^n < 3$; (b) $\left(1 + \frac{1}{n-1}\right)^n > \left(1 + \frac{1}{n}\right)^{n+1}$.

7. (**AM-GM Inequality**) Let $a_1, a_2, \cdots, a_n$ be any positive numbers, then $\sqrt[n]{a_1 a_2 \cdots a_n} \le \dfrac{a_1 + a_2 + \cdots + a_n}{n}$. Equality holds iff $a_1 = \cdots = a_n$.

8. Let $x > 0$ and $0 < \alpha < 1$, then $x^\alpha - \alpha x \le 1 - \alpha$.

9. Let a and b be positive numbers, and α, β be positive numbers satisfying $\alpha + \beta = 1$, then $a^\alpha b^\beta \le \alpha \cdot a + \beta \cdot b$.

10. (**Young's Inequality**) Let $a,\ b$ be positive numbers, and $p,\ q$ be positive numbers such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then $ab \le \dfrac{a^p}{p} + \dfrac{b^q}{q}$.

11. (**Hölder's Inequality**) Let n be a positive integer, $a_i > 0, b_i > 0$ $(i = 1, 2, \cdots, n)$, and $p,\ q$ be positive numbers such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then $\displaystyle\sum_{i=1}^n a_i b_i \le \left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n b_i^q\right)^{\frac{1}{q}}$. Equality holds iff $\dfrac{a_1^p}{b_1^q} = \dfrac{a_2^p}{b_2^q} = \cdots = \dfrac{a_n^p}{b_n^q}$.
    If $0 < p < 1$, then $\displaystyle\left(\sum_{i=1}^n a_i b_i\right) \ge \left(\sum_{i=1}^n a_i^p\right)^{1/p} \left(\sum_{i=1}^n b_i^q\right)^{1/q}$.

12. (**Minkowski's Inequality**) Let n be a positive integer, $k > 1$, and $a_i > 0, b_i > 0$ $(i = 1, 2, \cdots, n)$, then
    $$\left(\sum_{i=1}^n (a_i + b_i)^k\right)^{1/k} \le \left(\sum_{i=1}^n (a_i)^k\right)^{1/k} + \left(\sum_{i=1}^n (b_i)^k\right)^{1/k}.$$

13. (**Jensen's inequality**) Let $f : (a, b) \to \mathbb{R}$ be a convex function, then $f(\ q_1 x_1 + q_2 x_2 + \cdots + q_n x_n\ ) \le q_1 f(x_1) + q_2 f(x_2) + \cdots + q_n f(x_n)$, for all $q_i > 0$ satisfying $q_1 + q_2 + \cdots + q_n = 1$, and $x_i \in (a, b)$ $(i = 1, 2, \cdots, n)$.

14. (**Cauchy Inequality**) Let $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n$ be real numbers, then $\displaystyle\left(\sum_{i=1}^n x_i y_i\right)^2 \le \left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n y_i^2\right)$.

15. (**Power Mean inequality**) Let $x_1, x_2, \cdots, x_n$ be non-negative numbers, and $\alpha_1, \alpha_2, \cdots, \alpha_n$ be positive number so that $\alpha_1 + \cdots + \alpha_n = 1$, then $\displaystyle(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}) \le \sum_{i=1}^n \alpha_i x_i$. Equality holds if and only if $x_1 = \cdots = x_n$.

16. (**Rearrangement Inequality**) Let $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$ be real numbers. If π is a permutation π of the set $\{1, 2, \cdots, n\}$, then $\displaystyle\sum_{j=1}^n a_j \cdot b_{n-j} \le \sum_{j=1}^n a_j \cdot b_{\pi(j)} \le \sum_{j=1}^n a_j \cdot b_j$.
    Equality holds if and only if $a_1 = a_n$ or $b_1 = b_n$.

17. (**Chebyshev Inequality**) Let $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$ be real numbers, then $n\sum_{j=1}^{n} a_j b_{n-j} \le \left(\sum_{j=1}^{n} a_j\right)\left(\sum_{j=1}^{n} a_j\right) \le n\sum_{j=1}^{n} a_j b_j$.
Equality holds if and only if $a_1 = a_n$ or $b_1 = b_n$.
Hint: First prove that $\sum_{j=1}^{n}\sum_{k=1}^{n}(a_j - a_k)(b_j - b_k) \ge 0$.

1.2 Homework

1. Prove that S is bounded above if and only if $-S$ is bounded below.

2. Give example that S is bounded above but not bounded below.

3. Suppose that $(x_j)_{1\le j\le k}$ and $(y_j)_{1\le j\le k}$ are two finite sequence of complex numbers, and α and $\beta \in \mathbb{C}$. Using the definition of summation Σ and induction, prove that

   (a) $\sum_{j=1}^{k}(\alpha x_j + \beta y_j) = \alpha\sum_{j=1}^{k} x_j + \beta\sum_{j=1}^{k} y_j$.

   (b) If $x_j,\ y_j \in \mathbb{R}$ and $x_j \le y_j$ for all $1 \le j \le k$, then $\sum_{j=1}^{k} x_j \le \sum_{j=1}^{k} y_j$.

4. Let S be an non-empty subset of $\mathbb{Z}$, prove that

   (a) If S is bounded above, then $\min S \in S$.

   (b) If S is bounded below, then $\max S \in S$.

5. Suppose A and B are non-empty subsets of $\mathbb{R}$, define $A+B = \{\ x+y \mid x \in A \text{ and } y \in B\ \}$.

   (a) If $A \subset B$, prove that (i) $\sup A \le \sup B$ and (ii) $\inf A \ge \inf B$.

   (b) Prove: (i) $\sup(A + B) = \sup A + \sup B$, if one of them is finite; (ii) $\inf(A + B) = \inf A + \inf B$, if one of them is finite.

6. Prove that the complex field $\mathbb{C}$ cannot be ordered, i.e. there does not exist any non-empty subset P playing the same role of positive numbers in $\mathbb{R}$.

7. Let $a,\ b,\ c,\ d$ be rational numbers, and x is an irrational number such that $cx + d \ne 0$. Prove that $\dfrac{ax+b}{cx+d}$ is irrational if and only if $ad - bc \ne 0$.

8. (a) If $x, y \in \mathbb{R}$ then $2xy \le x^2 + y^2$ and $4xy \le (x+y)^2$. Equalities hold if and only if $x = y$.

   (b) If $a,\ b$ are positive real numbers, and $a + b = 1$, then $(a + 1/a)^2 + (b + 1/b)^2 \ge 25/2$. When does the equality hold?

   (c) If $a_1, a_2, \cdots, a_n$ are all positive real numbers, then $\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{j=1}^{n}\frac{1}{a_j}\right) \ge n^2$, and equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

   (d) If $a,\ b,\ c$ are positive real numbers and $a + b + c = 1$, then $\left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)\left(\frac{1}{c} - 1\right) \ge 8$, and equality holds if and only if $a = b = c = 1/3$.

   (e) If $a,\ b,\ c$ are positive real numbers, prove that $\left(\frac{a}{2} + \frac{b}{3} + \frac{c}{6}\right)^2 \le \frac{a^2}{2} + \frac{b^2}{3} + \frac{c^2}{6}$, and equality holds if and only if $a = b = c$.

   (f) If $a_1, a_2, \cdots, a_n$ and $w_1, w_2, \cdots, w_n$ are all positive real numbers with $\sum_{j=1}^{n} w_j = 1$. Prove that $\left(\sum_{j=1}^{n} a_j w_j\right)^2 \le \sum_{j=1}^{n} a_j^2 w_j$, and equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

9. Prove that $\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n} \le \frac{1}{\sqrt{3n+1}}$, and equality holds if and only if $n = 1$.

10. (a) For all $n \in \mathbb{N}$ we have $\sqrt{n+1} - \sqrt{n} < \frac{1}{2\sqrt{n}} < \sqrt{n} - \sqrt{n-1}$.
    Hint: $(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n}) = 1$.

(b) If k (> 1) is a positive integer, then $2\sqrt{k+1}-2 < \sum_{n=1}^{k} \frac{1}{\sqrt{n}} < 2\sqrt{k}-1$.

11. Let n be a positive integer, and $x \in \mathbb{R}$. Prove the following holds.

(a) If $-1 < x < 0$, then $(1+x)^n \leq 1 + nx + \frac{n(n-1)}{2}x^2$.

(b) If $x > 0$, then $(1+x)^n \geq 1 + nx + \frac{n(n-1)}{2}x^2$.

Hint: Compare Bernoulli's Inequality.

12. Prove that any positive rational r can be expressed in exactly one way in the form $r = \sum_{j=1}^{n} \frac{a_j}{j!}$, where $a_1, a_2, \cdots, a_n$ are integers such that $a_1 \geq 0$, $0 \leq a_j < j$ for $2 \leq j \leq n$, and $a_n \neq 0$.

13. Show that $n! \leq \left(\frac{n+1}{2}\right)^n$.

14. Find the infimum and supremum of the set $S = \{2^{-k} + 3^{-m} + 5^{-n} \mid k, m, n \text{ are positive integers } \}$.

15. If $a, b, c \in \mathbb{C}$ such that $|a| = |b| = |c|$ and $a + b + c = 0$, show that $|a-b| = |b-c| = |c-a|$. What is the geometrical meaning?

16. For any complex numbers $a, b \in \mathbb{C}$, show that $|a+b|^2 + |a-b|^2 = 2(|a|^2 + |b|^2)$.

17. For any complex numbers $a, b \in \mathbb{C}$ such that $\mathrm{Re}(\overline{a} \cdot b) = 0$, show that show that $|a-b|^2 = |a|^2 + |b|^2$. What is the geometrical meaning?

18. If $x, y \in \mathbb{R}$, and n is a positive integer, prove the following holds:

(a) $[x+y] \geq [x] + [y]$.

(b) $\left[\frac{[x]}{n}\right] = \left[\frac{x}{n}\right]$.

(c) $\sum_{k=0}^{n-1} \left[x + \frac{k}{n}\right] = [nx]$, where $[t]$ is the integral part of t.

1.3 Harder Exercises

1. Let $r_1, r_2 \in \mathbb{Q}$, define a sequence as follows: for any integer $n \geq 2$, we have $r_{n+1} = \dfrac{r_n + r_{n-1}}{2}$. Prove that (i) all the terms in $\{r_n\}_{n\geq 1}$ are rational; (ii) $\{r_n\}_{n\geq 1}$ is a Cauchy sequence.

2. For any given real number x, prove that there exists a unique integer n such that $n \leq x < n+1$. In this case, n is usually denoted by $[x]$, called the *integral part* of x.

3. Let E be the set of all Cauchy sequences of rational numbers. Suppose that K is a non-empty subset of E, i.e. K is a family of Cauchy sequences of rational numbers. K is called an *ideal* of E, if the following two conditions are satisfied:

(i) For any two sequences $\{r_n\}$ and $\{s_n\}$, the sequence $\{r_n+s_n\}_{n\geq 1} \in K$;

(ii) For any two sequences $\{r_n\}$ and $\{s_n\}$, the sequence $\{r_n \cdot s_n\}_{n\geq 1} \in K$.

K is called *maximal ideal* of E, if it satisfies the following two conditions: (i) K is an ideal of E; (ii) any ideal containing K is E or K.

(a) Prove that the set $K = \{ \{r_n\} \in E \mid \lim_{n\to+\infty} r_n = 0 \}$ is an ideal of E.

(b) If A is an ideal of E such that $K \subset A$ and $K \neq A$. Let $\{s_n\} \in A \setminus K$. Prove that

i. There exists $\{r_n\} \in E$ such that $\{r_n + s_n\} \in A$ and for all $n \in \mathbb{N}$, $r_n + s_n \neq 0$ and $\{\frac{1}{r_n+s_n}\} \in E$.

ii. The constant sequence $\{1_n\} \in A$, where $1_n = 1$ for all $n \in \mathbb{N}$.

iii. K is a maximal ideal of K.

4. Determine the set of cluster points of the sets $A = \{ \dfrac{1}{n} + \dfrac{1}{m} \mid n, m \in \mathbb{N} \}$ and $B = \{ \dfrac{m}{nm+1} \mid n, m \in \mathbb{N} \}$.

1.4 Exercises of Mathematical Induction

1. Establish the following formula for all $n \in \mathbb{N}$, by means of Mathematical Induction Principle.

   (a) $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$;

   (b) $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$;

   (c) $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$;

   (d) $\sum_{k=1}^{n} (\ a + (k-1)b\ ) = \frac{n(2a+(n-1)d)}{2}$;

   (e) $\sum_{k=1}^{n} a \cdot r^k = a\frac{r^n - 1}{r-1}$, where $r \neq 1$.

2. If $r > -1$ is a real number, use the mathematical Induction Principle t show that Bernoulli's inequality holds: $(1+r)^n \geq 1 + nr$ for any $n \in \mathbb{N}$.

3. Using Bernoulli's inequality to prove the following statements:

   (a) If $a > 1$ is a real number, then $a^n \geq a$ for all $n \in \mathbb{N}$.

   (b) If $0 < a < 1$ is a real number, then $0 < a^n \leq a$ for all $n \in \mathbb{N}$.

4. (i) If $0 < a < 1$, prove that $0 < a < \sqrt{a} < 1$.
   (ii) If $a > 1$, prove that $1 < \sqrt{a} < a$.

1. Let $f : [a, b] \to \mathbb{R}$ be an increasing function. Show that
   (i) f is injective, i.e. if $f(x) = f(y)$, then $x = y$.
   (ii) If $D = f[\ [a, b]\ ]$, then inverse function $f^{-1} : D \to [a, b]$ is increasing.

2. Let $f : D \to \mathbb{R}$ be a bounded function, E be a non-empty subset of D. Prove the following:
   (i) $\inf\{f(x) \mid x \in D\ \} \leq \inf\{\ f(x) \mid x \in E\ \}$;
   (ii) $\sup\{f(x) \mid x \in D\ \} \geq \inf\{\ f(x) \mid x \in E\ \}$.

3. Let $\{\ x_n\ \}$ be a bounded sequence, and define $y_n = \sup\{\ x_k \mid k = n, n+1, \cdots\ \}$ and $x_n = \inf\{\ x_k \mid k = n, n+1, \cdots\ \}$ for each $n \in \mathbb{N}$. Verify that
   (i) the sequence $\{y_n\}$ is bounded and non-increasing, and
   (ii) the sequence $\{z_n\}$ is bounded and non-decreasing.

4. Show that the following sequence $\{x_n\}$ is monotone (non-decreasing or non-increasing) and bounded, respectively:

   (a) $x_1 > 1$ and $x_{n+1} = 2 - x_n^{-1}$;

   (b) $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2 + x_n}$;

   (c) $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2x_n}$;

   (d) $x_1 = 1$ and $x_{n+1} = (2x_n + 3)/4$.

5. Let $0 < a_1 < b_1$ and $a_{n+1} = \sqrt{a_n b_n}$ and $b_{n+1} = \frac{a_n + b_n}{2}$ for each $n \in \mathbb{N}$.
   (i) Prove, by mathematical induction, that $a_n < b_n$ for every $n \in \mathbb{N}$.
   (ii) Prove that both $\{a_n\}$ and $\{b_n\}$ are monotone and bounded sequences.

6. (i) Modify the argument given in the example to show that there exists a real number, denoted by $\sqrt{3}$, satisfying $(\sqrt{3})^2 = 3$.
   (ii) Prove that the real number $\sqrt{3}$ is an irrational number.

7. Let $f : D \to \mathbb{R}$ be a bounded function, $a \in \mathbb{R}$ Show that

   (a) $\sup\{\ a + f(x) \mid x \in D\ \} = a + \sup\{\ f(x) \mid x \in D\ \}$.

   (b) $\inf\{\ a + f(x) \mid x \in D\ \} = a + \inf\{\ f(x) \mid x \in D\ \}$.

8. Let $f,\ g : D \to \mathbb{R}$ be two bounded functions with domain D. Show that

   (a) $\inf\{\ f(x) \mid x \in D\ \} + \inf\{\ g(x) \mid x \in D\ \} \leq \inf\{\ f(x) + g(x) \mid x \in D\ \}$.

   (b) $\sup\{\ f(x) \mid x \in D\ \} + \sup\{\ g(x) \mid x \in D\ \} \geq \inf\{\ f(x) + g(x) \mid x \in D\ \}$.

9. Let $f, g : D \to \mathbb{R}$ be functions, show that

$$\begin{aligned} \inf\{\ f(x) + g(x) \mid x \in D\ \} &\leq \inf\{\ f(x) \mid x \in D\} + \sup\{\ g(x) \mid x \in D\ \} \\ &\leq \sup\{\ f(x) + g(x) \mid x \in D\ \} \end{aligned}$$

2 Supremum and Infimum

2.1 Examples

1. Find the solution sets:
   (i) $S_1 = \{ x \in \mathbb{R} \mid |x - a| \le 2 \}$; and (ii) $S_2 = \{ x \in \mathbb{R} \mid |x^2 - a^2| \le 1$.
   And represent the solution S_i $(i = 1, 2)$ in terms of unions of intervals.

2. Is the set $S = \{ \dfrac{x+1}{x+3} \in \mathbb{R} \mid x \in (0,1) \}$ an interval? Give your reason.

3. Find the solution set S of the inequality: $-7 - 3x < 5x + 29$.

4. Find the solution set S of the inequality: $\dfrac{2x-3}{x+2} \le \dfrac{1}{3}$. And hence, prove that S is bounded, and determine $\sup S$ and $\inf S$.

5. Let $K = \{ 1/n \in \mathbb{R} \mid n = 1, 2, \cdots \} \cup \{0\}$. Prove that K is compact directly from the definition, without using Heine-Borel theorem.

6. Give an example of an open cover of the segment $(0,1)$ which has no finite subcover.

7. (a) If A and B are disjoint closed sets in some metric space X, prove that they are separated.

   (b) Prove the same for disjoint open sets.

   (c) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p,q) < \delta$, and define B similarly, with $>$ in place of $<$. Prove that A and B are separated.

   (d) Prove that every connected metric space with at least two points is uncountable. Hint Use (c).

2.2 Homework

1. Let $S = \{ 1 - (-1)^n/n \mid n = 1, 2, \cdots \}$. Find (i) $\sup S$ and (ii) $\inf S$.

2. Show in detail that the set $A = [0, +\infty)$ has lower bounds but no upper bounds.

3. Let $S \subset \mathbb{R}$ such that $\sup S \in S$. If $u \notin S$, show that $\sup(S \cup \{s\}) = \max(s, \sup S)$.

4. Show that a non-empty finite set S of $\mathbb{R}$ contains its supremum and infimum, i.e. $supS \in S$ and $\inf S \in S$. (Hint: Use induction.) Does the converse hold?

5. Let S be a non-empty subset of $\mathbb{R}$. Show that $u \in \mathbb{R}$ is an upper bound of S if and only if the following is satisfied: for any real number t, if $t < u$, then $t \notin S$.

6. Let S be a non-empty subset of $\mathbb{R}$. Show that $u = \sup S$ if and only if for every positive integer n, the number $u - 1/n$ is not an upper bound of S but the number $u + 1/n$ is an upper bound of S.

7. Suppose that A and B are bounded subsets of $\mathbb{R}$, prove that $A \cup B$ is bounded and that $\sup(A \cup B) = \sup\{\sup A, \sup B\}$.

8. Give an example of a countable collection of bounded subsets of $\mathbb{R}$ where (i) the union is bounded, and one where (ii) the union is unbounded.

9. Let S be a bounded set in $\mathbb{R}$ and let S_0 be a non-empty subset of S. Show that $\inf S \le \inf S_0 \le \sup S_0 \le \sup S$.

10. Let $a, b \in \mathbb{R}$ and S be a non-empty bounded set in $\mathbb{R}$. Let $aS = \{ as \mid s \in S \}$.

    (a) If $a > 0$, prove that $\inf(aS) = a \inf S$ and $\sup(aS) = a \sup S$.

    (b) If $a < 0$, prove that $\sup(aS) = a \inf S$ and $\inf(aS) = a \sup S$.

11. Let A and B be two bounded subset of $\mathbb{R}$. Let $A + B = \{ a + b \in \mathbb{R} \mid a \in A \text{ and } b \in B \}$. Prove that (i) $\sup(A+B) = \sup A + \sup B$;
    and (ii) $\inf(A+B) = \inf A + \inf B$.

12. Let X be a non-empty set and let $f : X \to \mathbb{R}$ be a function with bounded range, i.e. $\operatorname{ran}(f)$ is a bounded subset of R. Let $a \in \mathbb{R}$, show that (i) $\sup\{ a + f(x) \mid x \in X \} = a + \sup\{ f(x) \mid x \in X \}$; (ii) $\inf\{ a + f(x) \mid x \in X \} = a + \inf\{ f(x) \mid x \in X \}$.

13. Let X be a non-empty set, $f,\ g : X \to \mathbb{R}$ be two functions with bounded ranges in $\mathbb{R}$. Show that
(i) $\sup\{ f(x)+g(x) \mid x \in X \} \le \sup\{ f(x) \mid x \in X \}+\sup\{ g(x) \mid x \in X \}$;
(ii) $\inf\{ f(x)+g(x) \mid x \in X \} \ge \inf\{ f(x) \mid x \in X \}+\inf\{ g(x) \mid x \in X \}$.

14. Let $X = Y = (0, 1)$ be the unit interval in $\mathbb{R}$. Define $h : X \times Y \to \mathbb{R}$ by $h(x, y) = 2x + y$. Find

   (a) $f(x) = \sup\{ h(x, y) \mid y \in Y \}$, and $\inf\{ f(x) \mid x \in X \}$.

   (b) $g(x) = \inf\{ h(x, y) \mid y \in Y \}$, and $\sup\{ g(x) \mid x \in X \}$.

   Compare the results obtained in both part.

15. Given any $x \in \mathbb{R}$ show that there exists a unique integer n such that $n - 1 \le x < n$.

16. If $y > 0$ show that there exist a natural number n such that $1/2^n < y$.

17. Modify the argument given in the notes to show that

   (a) if $a > 0$, then there exists a positive real number z such that $z^2 = a$.

   (b) if $a > 0$ and any positive integer n , then there exists a positive real number z such that $z^n = a$.

18. Prove that Q is dense in $\mathbb{R}$.

19. If $u > 0$ and $x < y$, show that there exists a rational number r such that $x < ru < y$. Hence the set $\{ ru \mid r \in \mathbb{Q} \}$ is dense in $\mathbb{R}$.

2.3 Limsup and Liminf of Bounded Sequences

1. **Theorem**. A sequence (x_n) is called *contractive* if there exists a constant C with $0 < C < 1$ such that $|x_{n+2} - x_{n+1}| \le C|x_{n+1} - x_n|$ for all $n \in \mathbb{N}$. The number C is called the constant of the contractive sequence. Prove that contractive sequence is Cauchy, and hence is convergent.

2. **Stolz's Theorem**. Let (y_n) be a strictly increasing sequence, and $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = 0$. If $\lim\limits_{n\to\infty} \dfrac{x_n - x_{n+1}}{y_n - y_{n+1}}$ exists (finite or $\pm\infty$), then $\lim\limits_{n\to\infty} \dfrac{x_n}{y_n}$ exists and $\lim\limits_{n\to\infty} \dfrac{x_n}{y_n} = \lim\limits_{n\to\infty} = \dfrac{x_n - x_{n+1}}{y_n - y_{n+1}}$.

3. (**Limit Point**) Let $E \subset \mathbb{R}$, then the followings are equivalent:

   (a) a is an accumulation point (limit point) of the set E.

   (b) For any given $\varepsilon > 0$, the open interval $(a-\varepsilon, a+\varepsilon)$ contains infinitely many points of E.

   (c) For any given $\varepsilon > 0$, the punctured open interval $(a-\varepsilon, a+\varepsilon)$ contains at least a point of E.

   (d) There exists a sequence $\{x_n\} \subset E$ such that $x_n \ne x_m$ whenever $n \ne m$, and that $\lim\limits_{k\to\infty} x_n = a$.

4. **Definition**. Given a bounded sequence (x_n), let $A = \{ x_n \mid n = 1, 2, \cdots \}$ be the set of all the values taken by the terms of (x_n). Define $\overline{\lim\limits_{n\to\infty}}\ x_n = \overline{\lim}\ x_n = \sup A$ and $\underline{\lim\limits_{n\to\infty}}\ x_n = \underline{\lim}\ x_n = \inf A$. Then we have

   (a) For any $\varepsilon > 0$, there exists infinitely many n such that $x_n > \overline{a} - \varepsilon$.

   (b) For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $x_n < \overline{a} + \varepsilon$ for all $n \ge N$.

   (c) For any $\varepsilon > 0$, there exists infinitely many n such that $x_n < \underline{a} + \varepsilon$.

   (d) For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $x_n > \underline{a} - \varepsilon$ for all $n \ge N$.

5. Let $E \subset \mathbb{R}$, $\beta \notin E$. Then $\beta = \sup E$ if and only if it any one of the following conditions is satisfied:

(i) $x < \beta$ for all $x \in E$;

(ii) There exists an increasing sequence $(\ x_n\ )$ such that $\lim\limits_{n\to\infty} x_n = \beta$.

6. Let (x_n) be a bounded sequence. Prove that the following are equivalent:[1]

   (a) $\beta = \lim\limits_{n\to\infty} \sup\{\ x_k \mid k \geq n\ \}$

   (b) For any $\varepsilon > 0$, there are only finitely many terms of the sequence (x_n) greater than $\beta + \varepsilon$, and there are infinitely many terms of the sequence (x_n) greater $\beta - \varepsilon$.

   (c) There exist a subsequence (x_{n_k}) of the sequence (x_n) with $\lim\limits_{n\to\infty} x_{n_k} = \beta$, and for any convergent subsequence (x_{j_k}) of (x_n) with limit β', we have $\beta' \leq \beta$.

7. Suppose that (x_n), (y_n) are bounded sequence, prove that

   (a) (i) $\underline{\lim}\,(-x_n) = -\overline{\lim}\,x_n$; (ii) $\overline{\lim}\,(-x_n) = -\underline{\lim}\,x_n$;

   (b) For any subsequence (x_{n_k}) of (x_n), we have

   (i) $\underline{\lim}\limits_{k\to\infty}\,(x_{n_k}) \leq \underline{\lim}\,x_n$ (ii) $\overline{\lim}\,(x_{n_k}) \leq \overline{\lim}\,x_n$

   (c) If $x_n \leq y_n$ for all $n \in \mathbb{N}$, then

   (i) $\underline{\lim}\limits_{k\to\infty}\ x_n \leq \underline{\lim}\limits_{k\to\infty}\ y_n$, and (ii) $\underline{\lim}\limits_{k\to\infty}\ x_n \leq \underline{\lim}\limits_{k\to\infty}\ y_n$.

   (d) $$\underline{\lim}_{k\to\infty}\ x_n + \underline{\lim}_{k\to\infty}\ y_n \leq \underline{\lim}_{k\to\infty}\ (x_n + y_n) \leq \begin{cases} \underline{\lim}\limits_{k\to\infty}\ x_n + \overline{\lim}\limits_{k\to\infty}\ y_n; \\ \overline{\lim}\limits_{k\to\infty}\ x_n + \underline{\lim}\limits_{k\to\infty}\ y_n. \end{cases}$$

   (e) If $x_n \geq 0$ and $y_n \geq 0$ for all $n \in \mathbb{N}$, then

   $$\underline{\lim}_{k\to\infty}\ x_n \cdot \underline{\lim}_{k\to\infty}\ y_n \leq \underline{\lim}_{k\to\infty}\ (x_n \cdot y_n) \leq \begin{cases} \underline{\lim}\limits_{k\to\infty}\ x_n \cdot \overline{\lim}\limits_{k\to\infty}\ y_n; \\ \overline{\lim}\limits_{k\to\infty}\ x_n \cdot \underline{\lim}\limits_{k\to\infty}\ y_n, \end{cases}$$

   and $\underline{\lim}\limits_{k\to\infty}\ x_n \cdot \underline{\lim}\limits_{k\to\infty}\ y_n \leq \overline{\lim}\limits_{k\to\infty}\ (x_n \cdot y_n) \leq \overline{\lim}\limits_{k\to\infty}\ x_n \cdot \overline{\lim}\limits_{k\to\infty}\ y_n$.

[1] 数学分析：学习指导

8. Suppose that (x_n) is a sequence such that $0 \leq x_{n+m} \leq x_n + x_m$ for all $n,\ m \in \mathbb{N}$.

   (a) Prove that the sequence $\{\ \dfrac{x_n}{n}\ \}$ converges.

   (b) Prove that the sequence $\{\ x_n\ \}$ converges.

9. Suppose that $\{\ x_n\ \}$ is a bounded sequence, and that for any given $\varepsilon > 0$ and $n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $x_n < x_m + \varepsilon$ for all $n \geq N$. Prove that $\{\ x_n\ \}$ converges.

10. (a) Let $\{x_n\}$ be a sequence of positive numbers. If $\overline{\lim}\limits_{k\to\infty}\ x_n.\ \overline{\lim}\limits_{k\to\infty}\ \dfrac{1}{x_n} = 1$, show that $\{x_n\}$ converges.

    (b) Let $x_1 > 0$ and define $x_{n+1} = 1 + \frac{1}{x_n}$ for all $n \geq 1$. Prove that $\{x_n\}$ converges and find its limit.

11. Suppose that $\{x_n\}$ be a bounded sequence, and $\lim\limits_{k\to\infty}(x_n + 2x_{2n}) = 1$, show that $\lim\limits_{k\to\infty} x_n = \frac{2}{3}$.

12. Let $\{\ x_n\ \}$ be a sequence, and suppose that three of its subsequences $\{x_{2k}\}$, $\{x_{2k+1}\}$, $\{x_{3k}\}$ converge. Prove that $\{\ x_n\ \}$ is convergent. [2]

13. Suppose that $\lim\limits_{n\to\infty} x_n = +\infty$, prove that the sequence has a minimum.

14. Let $\{x_n\}$ be a monotone sequence such that $\lim\limits_{n\to\infty} \dfrac{x_1 + x_2 + \cdots + x_n}{n} = a$. Prove that $\lim\limits_{n\to\infty} x_n = a$.

15. Let $x_1 = a > 0$, $x_{n+1} = \dfrac{a}{1 + x_n}$ for all $n \in \mathbb{N}$. Prove that $\{x_n\}$ converges and find its limit.

16. Suppose that $x_1 > \sqrt{a}$ where $a > 1$, and define $x_{n+1} = \dfrac{a + x_n}{1 + x_n}$ for all $n \in \mathbb{N}$. Prove that $\{x_n\}$ converges and find its limit.

[2] p.79 数学分析：学习指导

17. Let $x_1 = a,\ x_2 = b$ and $x_{n+1} = \dfrac{x_n + x_{n-1}}{2}$ for $n = 2, 3, \cdots$. Prove that $\{x_n\}$ converges and find its limit.

18. Let $x_1 = \log a$ ($a > 0$), and $x_{n+1} = x_n + \log(a - x_n)$ for $n \in \mathbb{N}$. Prove that $\{x_n\}$ converges and find its limit.

19. Suppose that sequence $\{x_n\}$ satisfies $0 < x_n < 1$ and $(1 - x_n)x_{n+1} > \frac{1}{4}$ for all $n \in \mathbb{N}$. Prove that $\{x_n\}$ converges and find its limit.

20. Suppose that $\{x_n\}$ is a bounded divergent sequence, prove that there exist two subsequences of $\{x_n\}$ converge to two distinct limits.

21. Let $\{x_n\}$ and $\{y_n\}$ be two sequences such that $y_{n+1} = x_n + ax_{n+1}$ for all $n \in \mathbb{N}$.

    (a) If $|a| > 1$, prove that $\{y_n\}$ converges if $\{y_n\}$ converges.

    (b) If $|a| \leq 1$, does the result above still hold?

22. Let $f : [0, 1] \to [0, 1]$ be a continuous function, and $x_1 \in [0, 1]$. Define $x_{n+1} = f(x_n)$ for all $n \in \mathbb{N}$. Prove that $\{x_n\}$ converges if and only if $\lim\limits_{k\to\infty}(x_{n+1} - x_n) = 0$.

3 Limit and Continuity

1. Let $x_0 \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) \geq 0$ for all $x \in \mathbb{R}$, and $\lim\limits_{x\to x_0} f(x) = A$. Prove that (i) $\lim\limits_{x\to x_0} \sqrt{f(x)} = \sqrt{A}$. (ii) If $A > 0$, then $\lim\limits_{x\to x_0} \dfrac{1}{(f(x))^2} = \dfrac{1}{A^2}$.

2. Let $f : [0, +\infty) \to \mathbb{R}$ be a uniformly continuous function. Prove that if $\alpha > 0$, then $\lim\limits_{x\to+\infty} \dfrac{f(x)}{x^{1+\alpha}} = 0$

3. Let $(\ A_n\ )_{n\in\mathbb{N}}$ be a family finite subset of $[0, 1]$, and that $A_n \cap A_m = \emptyset$. Define a function $f : [0, 1] \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x \in A_n \text{ for some } n \in \mathbb{N}; \\ 0 & \text{if } x \in [0,1] \setminus \bigcup_{n=1}^{\infty} A_n. \end{cases}$$

Show that for any $x_0 \in [0, 1]$, then $\lim\limits_{x\to x_0} f(x) = 0$.

4. Let $f : [a, b] \to \mathbb{R}$ be a strictly increasing function, and $\{x_n\}$ be a sequence such that $a < x_n < b$ for all $n \in \mathbb{N}$. If $\lim\limits_{n\to\infty} f(x_n) = f(a)$, prove that $\lim\limits_{k\to\infty} x_n = a$.

5. Suppose that $f : [a, b] \to \mathbb{R}$ be an unbounded function, prove that $\exists x_0 \in [a, b]$ such that the function f is unbounded on any neighborhood of x_0.

6. Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions, f is monotonic, and there exists a sequence $\{\ x_n\ \}_{n\geq 1} \subset [a, b]$ such that $g(x_n) = f(x_{n+1})$ for all $n \geq 1$. Prove that there exists $x_0 \in [a, b]$ such that $f(x_0) = g(x_0)$.

7. Let I be a bounded interval, $f : I \to \mathbb{R}$ is uniformly continuous if and only if $\{\ f(x_n)\ \}_{n\in\mathbb{N}}$ is Cauchy whenever $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in I. [3]

8. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be Lipschtiz, if there exists a constant $0 < L < 1$ such that $|f(x) - f(y)| \leq L|x - y|$, for any $x,\ y \in \mathbb{R}$. Prove that there exists a unique $x_0 \in \mathbb{R}$ such that $f(x_0) = x_0$.

3.1 Examples

1. Let a be the finite limit of the sequence $\{x_n\}$, prove that $\lim\limits_{n\to\infty} \dfrac{x_1 + x_2 + \cdots + x_n}{n} = a$. Does the conclusion hold if the limit a is not finite?

2. Suppose that $\{\ p_k\ \}$ is a sequence of positive real numbers, and that $\lim\limits_{n\to\infty} \dfrac{p_n}{p_1 + p_2 + \cdots + p_n} = 0$, and $\lim\limits_{n\to\infty} a_n = a$. Show that

$$\lim_{n\to\infty} \frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1 + p_2 + \cdots + p_n} a.$$

[3] p.86.数学分析：学习指导

3. Let $\{x_n\}$ be a sequence of real numbers such that $\lim_{n\to\infty}(x_n - x_{n-2}) = 0$. Prove that $\lim_{n\to\infty}\frac{x_n - x_{n-1}}{n} = 0$.

4. Starting from the first definition of limit, prove that $\lim_{n\to\infty}\sqrt{\frac{7}{16x^2-9}} = 1$.

5. Prove that the limit $\lim_{n\to\infty}\sin n$ does not exist.

6. Let x_0 be a real number, and I be a neighborhood of x_0 possibly not containing x_0, $f : I \to \mathbb{R}$ be a function defined on I satisfying the following condition:
If $\{x_n\}$ is a sequence in I such that $\lim_{n\to\infty} x_n = x_0$ and satisfying $0 < |x_{n+1} - x_0| < |x_n - x_0|$, then we have $\lim_{n\to\infty} f(x_n) = a$. Show that
$$\lim_{x\to x_0} f(x) = a.$$

7. Given any sequence $\{x_n\}$ of real numbers, prove that there exists a monotonic subsequence (but not necessarily strictly monotonic).

8. Establish the following limits by means of $\varepsilon - N$ definition:
(a) $\lim_{n\to\infty}\sqrt[n]{n} = 1$; (b) $n^3q^n = 0$ $(|q| < 1)$; (c) $\lim_{n\to\infty}\frac{\log n}{n^2} = 0$.

9. Suppose that $f(x)$, $g(x)$ are defined in some neighborhood, and that $g(x) > 0$, $\lim_{x\to 0}\frac{f(x)}{g(x)} = 1$. Suppose that $\{ a_{mn} \}$ is a double sequence of real numbers satisfying the following condition: $\forall\varepsilon > 0$, $\exists N(\varepsilon) > 0$ such that for all $n > N(\varepsilon)$ and $m = 1, 2, \cdots n$, we have $|a_{mn}| < \varepsilon$. Suppose that a_{mn} are all non-zero, prove that
$$\lim_{n\to\infty}\sum_{m=1}^{n} f(a_{mn}) = \lim_{n\to\infty}\sum_{m=1}^{n} g(a_{mn}),$$
as long as the right limit exists.

10. Show that $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\sqrt[3]{1+\frac{i}{n^2}} - 1\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{3n^2} = \frac{1}{6}$, and determine the limit $\lim_{n\to\infty}\sum_{i=1}^{n}(a^{\frac{i}{n^2}} - 1)$, for $a > 0$.

11. Suppose that $\{a_n\}_{n\geq 1}$ is a sequence of real numbers such that $\lim_{n\to\infty}\frac{a_1 + a_2 + \cdots + a_n}{n} = a < +\infty$, prove that $\lim_{n\to\infty}\frac{a_n}{n} = 0$.

12. Let $\{a_n\}$ be a sequence of positive numbers and there exists $C > 0$ such that $a_n \leq Ca_m$ for all $m < n$. Suppose that there exists a subsequence in $\{a_n\}$ converging to 0. Prove that $\lim_{n\to\infty} a_n = 0$.

3.2 Method of finding limit

1. **Elementary transformation**. One can use the elementary methods to transform or to simplify the analytic formula of a_n, and eventually obtain a more compact formula.

   (a) $x_n = \cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\cdots\cos\frac{x}{2^n}$.

   (b) $x_n = \frac{3}{2}\cdot\frac{5}{4}\cdot\frac{17}{16}\cdots\frac{2^{2n}+1}{2^{2n}}$.

   (c) $x_n = \sum_{i=1}^{n}\frac{1}{\sqrt{1^3 + 2^3 + \cdots + i^3}}$.

3.3 Easy Exercise

Let (x_n) and (y_n) be two sequences of real numbers. Suppose that $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to a} y_n = b$. Let $c \in \mathbb{R}$, and $k \in \mathbb{Z}$. Prove that

1. $\lim_{n\to\infty}(x_n + y_n) = a + b$;
2. $\lim_{n\to\infty}(x_n - y_n) = a - b$;
3. $\lim_{n\to\infty} cx_n = ca$;
4. $\lim_{n\to\infty} x_n \cdot y_n = ab$;
5. $\lim_{n\to\infty} x_n^k = a^k$;
6. $\lim_{n\to\infty} x_n/y_n = a/b$, if $b \neq 0$ and $y_n > 0$ for all $n \geq 1$.
7. $\lim_{n\to\infty}\sqrt{x_n} = \sqrt{a}$, if $a > 0$ and $x_n > 0$ for all $n \geq 1$.

4 Exercises of Rudin's PMA

4.1 The real and complex number systems

1. If r is a non-zero rational and x is irrational, prove that $r+x$ and $r \cdot x$ are irrational.

2. Prove that there is no rational number whose square is 12.

3. Let E be a non-empty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

4. Let A be a non-empty set of real numbers which is bounded below. Let $-A$ be the set of all number $-x$, where $x \in A$. Prove that $\inf A = -\sup(-A)$.

5. Fix $b > 1$.

   (a) If m, n, p, q are integers, $n > 0, q > 0$, and $r = m/n = p/q$, prove that $(b^m)^{1/n} = (b^p)^{1/q}$. Hence it makes sense to define $b^r = (b^m)^{1/n}$.

   (b) Prove that $b^{r+s} = b^r \cdot b^s$ if r and s are rational.

   (c) If x is real, define $B(x)$ to be the set b^t, where t is rational and $t \leq x$. Prove that $b^r = \sup B(r)$ when r is rational. Hence it makes sense to define $b^x = \sup B(x)$ for every $x \in \mathbb{R}$.

   (d) Prove that $b^{x+y} = b^x \cdot b^y$ for all real x and y.

6. Fix $b > 1$, $y > 0$, and prove that there is a unique real x such that $b^x = y$, by completing the following outline:

   (a) For any positive integer n, $b^n - 1 \geq n(b-1)$.

   (b) Hence $b - 1 \geq n(b^{1/n} - 1)$.

   (c) If $t > 1$ and $n > (b-1)/(t-1)$, then $b^{1/n} < t$.

   (d) If w is such that $b^w < y$, then $B^{w+1/n} < y$ for sufficiently large n; to see this, apply part (c) with $t = y \cdot b^{-w}$.

   (e) If $b^w > y$, then $b^{w-1/n} > y$ for sufficiently large n.

   (f) Let A be the set of all w such that $b^2 < y$, and show that $x = \sup A$ satisfying $b^x = y$.

   (g) Prove that this x is unique.

7. Prove that no order can be defined in the complex field that turns it into an ordered field. Hint: -1 is a square.

8. Suppose $z = a + bi$ and $w = c + di$. Define$z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. Does this ordered set have the least upper bound property?

4.2 Basic Topology

1. Prove that the emptyset $\emptyset$ is a subset of every set.

2. A complex number z is said to be *algebraic* if there are integers $a_0, a_1, \cdots, a_n$, not all zero, such that $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$. Prove that the set of all algebraic numbers is countable.
   Hint: For every positive integer N there are only finitely many equations with $n + |a_0| + |a_1| + \cdots + |a_n| = N$.

3. Prove that there exist real numbers which are not algebraic.

4. Is the set of all irrational real numbers countable?

5. Construct a bounded set of real numbers with exactly three limit points.

6. Let E' be the set of all limit points of a set $E \subset \mathbb{R}$. Prove that E' is closed. Prove that E and $\overline{E}$ have the asme limit points. (Recall the closure $\overline{E} = E \cup E'$.) Do E and E' always have the same limit points?

7. Let $A_1, A_2, A_3 \cdots$, be a sequence of subsets of a metric space (X, d).

   (a) If $B_n = \bigcup_{i=1}^n A_i$, prove that $\overline{B_n} = \bigcup_{i=1}^n \overline{A_i}$, for $i = 1, 2, 3, \cdots$.

(b) If $B_n = \bigcup_{i=1}^{\infty} A_i$, prove that $\overline{B_n} \supset \bigcup_{i=1}^{\infty} \overline{A_i}$. Show, by an example, that this conclusion can be proper.

8. Is every point of every open $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed set in $\mathbb{R}^2$.

9. Let E° denote the set of all interior points of a set E. Recall a point x is called an interior of E if there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset E$.

   (a) Prove that E° is always open.

   (b) Prove that E is open if and only if $E^\circ = E$.

   (c) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.

   (d) Prove that the complement of E° is the closure of the complement of E.

   (e) Do E and $\overline{E}$ always have the same interiors?

   (f) Do E and E° always have the same closures?

10. Let X be an infinite set. For any $p, q \in X$ define $d(p,q) = \begin{cases} 1 & \text{if } p \neq q; \\ 0 & \text{if } p = q. \end{cases}$
Prove that this is a metric on X. Which subset of the resulting metric space are open? Which are closed? Which are compact?

11. For $x, y \in \mathbb{R}^1$, define $d_1(x,y) = (x-y)^2$; $d_2(x,y) = \sqrt{|x-y|}$ $d_3(x,y) = |x^2 - y^2|$; $d_4(x,y) = |x - 2y|$; $d_5(x,y) = \dfrac{|x-y|}{1+|x-y|}$.
Determine, for each of these, whether it is a metric or not.

12. Let $K \subset \mathbb{R}^1$ consists of 0 and the number $1/n$, for $n = 1, 2, \cdots$. Prove that K is compact directly from the definition (without using Heine-Borel theorem).

13. Construct a compact set of real numbers whose limit points form a countable set.

14. Give an example of an open cover of the segment $(0,1)$ which has no finite subcover.

15. Show that the following theorem does not hold (in $\mathbb{R}^1$ for example) if the word "compact" is replaced by " closed" or by " bounded" alone.

    If $\{K_\alpha\}$ is a collection of compact subset of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is non-empty, then $\cap_\alpha K_\alpha$ is non-empty.

16. Let $\mathbb{Q}$ be the set of all rational numbers, define $d(p,q) = |p - q|$ for all $p, q \in \mathbb{Q}$. Let E be the set of all $p \in \mathbb{Q}$ such that $1 < p^2 < 3$. Show that E is closed and bounded in $\mathbb{Q}$, but that E is not compact. Is E open in $\mathbb{Q}$?

17. Let E be the set of all $x \in [0,1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0,1]$? Is E compact? Is E prefect?

18. Is there a non-empty prefect set in $\mathbb{R}^1$ which contains no rational numbers?

19. (a) If A and B are disjoint closed sets in some metric space, prove that they are separated.

    (b) Prove that the same for disjoint open sets.

    (c) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p,q) < \delta$, define B similarly, with $>$ in place of $<$. Prove that A and B are separated.

    (d) Prove that every connected metric space with at least two points is uncountable. Hint: Use (c).

5 Tests and Quizzes

5.1 Midterm and Final

1.1 Let x be a real number, and let n, m be natural numbers. Without using any of the exponent laws (other than the definition of exponentiation), show that $x^{n+m} = x^n \cdot x^m$. (Hint: mathematical induction.) [4]

1.2 Let $(a_n)_{n\geq 0}$ be a sequence of real numbers, such that $a_{n+1} > a_n$ for each natural number. Prove that $a_n > a_m$ for all $n > m$.

1.3 Let A, B be finite sets. Show that $A \cup B$ and $A \cap B$ are also finite sets, and $n(A) + n(B) = n(A \cup B) + n(A \cap B)$, where $n(X)$ denotes the number of elements in X.

1.4 Let $(a_n)_{n\geq 0}$ be a sequence of rational numbers which is bounded. Let $(b_n)_{n\geq 0}$ be another sequence of rational numbers which is equivalent to $(a_n)_{n\geq 0}$. Show that the sequence $(b_n)_{n\geq 0}$ is bounded.

1.5 Let $E \subset \mathbb{R}$ be non-empty, and suppose that E has a least upper bound M. Let $-E = \{ -x \in \mathbb{R} \mid x \in E \}$. Show that $\inf(-E) = -M$.

2.1 Let $(a_n)_{n\geq 0}$ be a sequence of real numbers which converges to 0, i.e. $\lim_{n\to\infty} a_n = 0$. Show that the series $\sum_n (a_n - a_{n+1})$ is conditionally convergent, and converges to a_0.
(Hint: First work out what the partial sums $\sum_{n=0}^{N}(a_n - a_{n+1})$ should be, and prove your assertion using induction.)

2.2 Let $\sum a_n$ be an absolutely convergent series of real numbers. Let $f : \mathbb{N} \to \mathbb{N}$ be an increasing function (i.e. $f(n+1) > f(n)$ for for all $n \in \mathbb{N}$). Show that $\sum_n a_{f(n)}$ is also an absolutely convergent series.
(Hint: try to compare each partial sum of $\sum_n a_{f(n)}$ with a (slightly different) partial sum of $\sum_n a_n$.)

2.3 A point x is called an adherent point of a subset S of $\mathbb{R}$ if for any $\varepsilon > 0$, the interval $(x - \varepsilon, x + \varepsilon) \cap S \neq \emptyset$. If S is bounded, show that $\sup E$ is an adherent point of E, and is also an adherent point of $\mathbb{R} \setminus E$.

2.4 Let X, Y, Z be subsets of $\mathbb{R}$. Let $f : X \to Y$ be a function which is uniformly continuous on X, and let $g : Y \to Z$ be a function which is uniformly continuous on Y. Show that the function $g \circ f : X \to Z$ is uniformly continuous on X.

2.5 Let $f : [0, 1] \to [0, 1]$ be a continuous function. Show that there exists a real number $c \in [0, 1]$ such that $f(c) = c$.
(Hint: Apply the intermediate value theorem to the function $f(x) = x$.)

3.1 Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function whose derivative $f' : \mathbb{R} \to \mathbb{R}$ is a bounded function. Show that f is uniformly continuous.
(Hint: use the mean-value theorem to get some sort of upper bound on $|f(x) - f(y)|$ for $x, y \in \mathbb{R}$.)

3.2 Let $\sum_n a_n$ be an absolutely convergent series of real numbers such that $\sum_{n=0}^{\infty} |a_n| = 0$. Show that $a_n = 0$ for every natural number n.

3.2 Let $f : [0, +\infty) \to \mathbb{R}$ be a non-negative, monotone decreasing function Suppose that there exists a number $M > 0$ such that $\int_{[0,N]} f(x)dx \leq M$ for all $N \in \mathbb{N}$. Show that the sum $\sum_{n=1}^{\infty} f(n)$ is convergent.
Hint: compare the sum $\sum_{n=1}^{N} f(n)$ and the integral $\displaystyle\int_{[0,N]} f(x)dx$?

3.4 Let X be a finite subset of $\mathbb{R}$. Show that $\overline{X} = X$, i.e. the closure of X is the same as X itself.

3.5 Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. Let $g : [-b, -a] \to \mathbb{R}$ be defined by $g(x) = f(-x)$. Show that g is also Riemann integrable and $\displaystyle\int_{[-b,-a]} g(x)dx = \int_{[a,b]} f(x)dx$.

[4] Taken from Tao's 131-A

3.6 Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a continuous, non-negative function (so $f(x) \geq 0$ for all $x \in [a, b]$). Suppose that $\int_{[a,b]} f(x)dx = 0$. Show that $f(x) = 0$ for all $x \in [a, b]$. (Hint: argue by contradiction.)

5.2 Tests and Quizzes

This is a closed book examination, no notes are allowed. You have 3 hours to complete it.[5]

1. Prove Rolle's Theorem: If f is a continuous on $[a, b]$ and differentiable on (a, b), with $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

2. Prove that if f is Lebesgue integrable on $[a, b]$, then so is $|f|$, and then $\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx$.

3. Prove the intermediate value theorem: If f is continuous on $[a, b]$ and $f(a) < f(b)$, then f takes every value between $f(a)$ and $f(b)$, i.e. for any $d \in [\ f(a), f(b)\ ]$ there exists $c \in [a, b]$ such that $f(c) = d$.

4. Let $f(x) = 3x^2 + 7x + 3$ be a function defined on $\mathbb{R}$. Given any $\varepsilon > 0$, find a (specific) $\delta > 0$ (depending on ε) such that $|f(x) - f(0)| < \varepsilon$ whenever $|x| < \delta$.

5. Determine with reason whether this series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^{(n+1)/n}}{n!}$ converges.

6. Show that there exists some constant K independent of n such that $0 < \dfrac{1}{n} - \dfrac{1}{\sqrt[3]{n^3+1}} \leq \dfrac{K}{n^4}$.

7. (a) Show that in any metric space, the limit of a convergent sequence is unique, i.e. if $x_n \to \alpha$ and $x_n \to \beta$, then $\alpha = \beta$.

   (b) Show that the following two definitions of continuity of a function $f : (X, \rho) \to (Y, \sigma)$ are equivalent:

   i. To any $x_0 \in X$ and $\varepsilon > 0$ there corresponds $\delta > 0$ such that $\rho(x, x_0) < \delta$ implies $\sigma(\ f(x), f(x_0)\ ) < \varepsilon$.

   ii. For any open subset V in (Y, σ), the inverse image $f^{-1}[V] = \{\ x \in X \mid f(x) \in V\ \}$ is open in (X, ρ).

8. Recall the definition $\limsup x_n = \lim_{n\to\infty} \sup\{\ x_k \mid k \geq n\ \}$. Prove the following:

   (a) If $\{x_n\}$, $\{y_n\}$ are bounded sequences, then $\liminf(\ x_n + y_n) \leq \liminf\ x_n + \limsup\ y_n$.

   (b) Give a counterexample to show that $\liminf(x_n + y_n) \leq \liminf x_n + \liminf y_n$ does not hold in general.

   Hint: There are several ways in which you can approach (a), you may use any of the following, (but you're not likely to need them all.)

   (i) $\limsup x_n$, $\liminf x_n$ exist whenever $\{x_n\}$ is a bounded sequence.

   (ii) $\sup(A + B) = \sup A + \sup B$, and $\sup(-A) = -\inf(A)$.

   (iii) For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $x_\leq \limsup x_n + \varepsilon$ whenever $k > N$, and $\liminf x_n - \varepsilon < x_k$ for some $k > N$,

In the following, please show your work. Answers without explanation will receive no credit. Each quiz is worth 8 points in total.

1.1 (4 points) Consider the following sentence: "Every natural is divisible by some other natural number."

   (a) Rewrite the above statement using the symbols $\forall$, $\exists, \Rightarrow$ where appropriate.

   (b) Negate your statement from part (a).

[5] 104f.tex

1.2 (4 points) Prove that $(2n+1)+(2n+3)+(2n+5)+\cdots+(4n-1)=3n^2$ for all positive integers n.

3.1 (5 points)

(a) Show that $\sqrt[3]{5}$ is not rational.

(b) Show that $2+\sqrt[3]{5}$ is not rational.

3.2 (3 points) Suppose that $r \in \mathbb{Q}$ and $x \notin \mathbb{Q}$. Show that $rx \notin \mathbb{Q}$.

4.1 (4.5 points) Let S and T be tow non-empty subsets of $\mathbb{R}$ such that for all $s \in S$ and $t \in T$, $s \leq t$.

(a) Prove that $\sup S$ and $\inf T$ exist.

(b) Prove that $\sup S \leq \inf T$.

4.2 (3.5 points) Prove (using the $\varepsilon - N$ definition of limit) that $\lim_{n\to\infty} \frac{2n^2+3}{n^2-7} = 2$.

5.1 (a) (2 points) Suppose s_n is a sequence of nonnegative real numbers that converges to zero. Prove that the sequence $\sqrt{s_n}$ also converges to zero.

(b) (2 points) Suppose s_n is a sequence of nonnegative real numbers that converges to $s \neq 0$. Prove that the sequence $\sqrt{s_n}$ converges to $\sqrt{s}$. Hint: Use the fact that $\sqrt{x}-\sqrt{y} = \dfrac{x-y}{\sqrt{x}+\sqrt{y}}$.

5.2 (a) (2 points) State the precise definition of $\lim_{n\to\infty} t_n = +\infty$.

(b) (2 points) Use the precise definition of $\lim_{n\to\infty} t_n = +\infty$ to prove that $\lim_{n\to\infty} \dfrac{3n^2+2n}{n+4} = +\infty$.

6.1 Determine whether the following limits exist and prove your assertions.

(a) (2.5 points) $\lim_{x\to 2} \dfrac{|x-2|}{x-2}$.

(b) (2.5 points) $\lim_{x\to 0} f(x)$, where $f(x) = \begin{cases} x^3 & \text{if } x \in \mathbb{R}\setminus\mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{Q}. \end{cases}$.

(c) (3 points) $\lim_{x\to -3^-} \dfrac{1}{(x+3)^5}$.

7.1 (3 points) For the following function, find the set of points of continuity and the set of points of discontinuity of function. Justify your answers in both cases. $f(x) = \begin{cases} \frac{1}{x}\cos\frac{1}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$

8.1 Determine whether the following continuous functions are uniformly continuous on the specified interval. You must justify your answers; you can use any theorems about uniform continuity presented in class.

(a) (2 points) $f(x) = \frac{1}{x-4}$ on the interval $(4, 6]$.

(b) (2 points) $g(x) = \frac{1}{x-4}$ on the interval $[6, +\infty)$.

(c) (2 points) $h(x) = \cos\frac{1}{x}$ on the interval $(0, \frac{1}{\pi}.)$

8.2 (2 points) Show that the function $f(x) = \begin{cases} x\cos\frac{1}{x} & \text{if } x \neq 0; \\ 0 & \text{if} x = 0 \end{cases}$ is not differentiable at $x = 0$. (You may use any results here that we've previously shown...)

9.1 Determine whether the following functions are integrable on the specified interval $[a, b]$. If the function is integrable, determine $\int_a^b f(x)dx$. You may use any relevant results that we have encountered.

(a) (3 points) $f(x)$ on $[1, 3]$, where $f(x) = \begin{cases} 2 & \text{if } x \in \mathbb{R}\setminus\mathbb{Q}; \\ 0 & \text{if } x \in \mathbb{Q}. \end{cases}$

(b) (2.5 points) $g(x)$ on $[0, 1]$, where $g(x) = \begin{cases} 1 & \text{if } x = \frac{1}{3} \text{ or } \frac{2}{3}; \\ 0 & \text{otherwise.} \end{cases}$

9.2 (2.5 points) Compute the following limit, if it exists: $\lim_{x\to\pi} \frac{\int_{\pi^2}^{x^2} e^t \sin\sqrt{t}\, dt}{x-\pi}$.

10.1 For each $n \in \mathbb{N}$, let $f_n(x) = \frac{\sin(n^2 x)}{n^2}$.

(a) (1.5 points) For each $x \in \mathbb{R}$, determine $\lim_{n\to\infty} f_n(x)$.

(b) (3 points) Does the sequence (f_n) converge uniformly on $\mathbb{R}$? Justify your answer.

(c) (2 points) Does the series $\sum_{n=1}^{\infty} f_n(x)$ converge uniformly on $\mathbb{R}$? Justify your answer.

(d) (1.5 points) Does the series $\sum_{n=1}^{\infty} f_n(x)$ represent a continuous function on $\mathbb{R}$? Justify your answer.

5.3 Sample Final Examinations

1. Let $f : A \to B$ be a function, show that f is injective (one-to-one) if and only if $f^{-1}[f[C]] = C$ for all $C \subset A$.

2. Let x_n be a sequence in a complete metric space (M, d) so that $d(x_n, x_{n+1}) \leq 1/n^2$. Does this imply that the sequence (x_n) converges?

3. Show that in a metric space any open set is a countable union of closed sets.

4. Let $\sum_n f_n$ be a series of functions which is uniformly convergent on $[a, b]$. Show that the series $\sum \frac{f_n}{n}$ also converges uniformly on $[a, b]$.

5. Let $f : [0, 1] \to \mathbb{R}$ be a continuously differentiable function in $[0, 1]$ with $f(0) = 0$ and $f(1) = 1$. Prove that $\in_0^1 |f'(x)|^2\, dx \geq 1$.

6. (a) State the definition of the limit of a function at a point.

   (b) Decide whether $\lim_{x\to 0} x \sin\frac{1}{x}$ exists. Prove your assertion.

7. Find the Taylor series at 0 for the function $f(x) = \frac{3}{(1-x)(1+2x)}$. Decide whether and where the series converges to the function f.

5.4 Problems

1. (a) Determine whether each of the following series converges. In each case, carefully justify your answer, stating any results you use: (i) $\sum \frac{n}{2^n}$, (ii) $\sum \frac{n^2 + (-1)^n n}{n^4 + n^3 + \sqrt{n}}$.

   (b) Show that if $p,\ q > 0$ then the series $\sum (-1)^n \frac{(\ \log(n+1)\ )^p}{(n+1)^q}$ converges.

   (c) Prove that $uv \leq (u^2 + v^2)/2$ for any real numbers u and v. Suppose that the series $\sum_n a_n$ is convergent, and that $a_n \geq 0$ for all $n \in \mathbb{N}$. Prove that the series $\sum \sqrt{a_n a_{n+1}}$ is also convergent.

2. (a) Determine the set of all $x \in \mathbb{R}$ for which the series $\sum_n \frac{x^{3n}}{2^n \sqrt{n}}$ converges. For which x is it absolutely convergent? For which is it conditionally convergent?

   (b) Determine whether the series $\sum a_n$ converges, where $a_n = \frac{n!}{3\cdot 5\cdot 7\cdots(2n+1)}$.

   (c) Suppose that $\sum a_n$ is a series of strictly positive terms and define b_n by $b_n = \frac{s_n}{n}$, where s_n is the nth partial sum of $\sum a_n$. Prove that if t_n is the nth partial sum of $\sum b_n$ then $t_n \geq a_1 r_n$ where r_n is the nth partial sum of the harmonic series $\sum 1/n$. Hence prove that $\sum b_n$ always diverges.

3. (a) State the Mean Value Theorem.

   Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Suppose also that $f(a) < 0 < f(b)$ for some real numbers $a < b$, and htat $0 < m \leq f'(x) \leq M$ for all $x \in \mathbb{R}$. Given $x_\in (a, b)$, define the sequence (x_n) by $x_{n+1} = x_n - \frac{f(x_n)}{M}$. why is there is a unique solution $c \in (a, b)$ of the equation $f(x) = 0$? Prove that $|x_{n+1} - c| \leq \frac{|f(x_1)|}{m}\left(1 - \frac{m}{M}\right)^n$, and deduce that $x_n \to c$ as $n \to \infty$.

You may find it useful to note that
$x_{n+1} = x_n - \frac{f(x_n)}{M} = x_n - \frac{f(x_n) - f(c)}{M}.$

(b) Suppose that for a constant α, the function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by
$$g(x, y) = \begin{cases} \frac{x^\alpha}{x^2+y^2} & \text{if } (x, y) \neq (0, 0); \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$
For which values of α is the function continuous at $(x, y) = (0, 0)$? Justify your answer.

4. (a) State the Bolzano-Weierstrass Theorem for sequence of real numbers. Suppose I is the interval $[a, b]$ with $b > a$. and that $f : I \to \mathbb{R}$ is continuous with the property that for each $x \in I$, there is $y \in I$ such that $|f(y)| \leq |f(x)|/2$. Prove that there exists $c \in I$ such that $f(c) = 0$, stating clearly any results you use.

(b) What does it mean to say that a function $g : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}^2$?

What is meant by the *directional derivative* of $g : \mathbb{R}^2 \to \mathbb{R}$ at $a \in \mathbb{R}^2$?

Suppose that $g : \mathbb{R}^2 \to \mathbb{R}$ is defined by
$$g(x, y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if } (x, y) \neq (0, 0); \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$
Prove that g has directional derivative at $(x, y) = (0, 0)$ in all directions, and determine these. Prove, however that g is not differentiable at $(0, 0)$.

5. (a) What is meant by an open subset of $\mathbb{R}^m$? What is meant by a closed subset of $\mathbb{R}^m$? Prove that if U is an open subset of $\mathbb{R}^m$ then its complement $\mathbb{R}^m \setminus U$ is a closed set.

(b) What does it mean to say that a subset C of $\mathbb{R}^m$ is connected? Prove that if C is a connected subset of $\mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}^n$ is continuous, then $f[C] = \{ f(x) \mid x \in C \}$ is a connected subset of $\mathbb{R}^n$.

(c) What is meant by a compact subset of $\mathbb{R}^m$?

Suppose that for each natural number n, C_n is a non-empty compact subset of $\mathbb{R}^m$ and that $C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$ i.e. $C_{k+1} \supseteq C_k$ for each k. Prove that $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$.

Hint: consider any sequence (x_n) in which $x_n \in C_n$.

6. (a) Define what it means to say that (A, d) is a metric space. What is meant by an *open subset* of (A, d)?

Suppose (A, d) is a metric space. Define $d_2 : A \times A \to \mathbb{R}$ by $d_2(x, y) = \frac{d(x, y)}{1 + d(x, y)}$. Verify that d_2 is a metric on A.

(b) Suppose that (A_1, d_1), (A_2, d_2) are metric spaces and that f is a function mapping from A_1 to A_2. What does it mean to say that f is (d_1, d_2)-continuous?

Suppose that (A, d) is a metric space. Let $\emptyset \neq B \subset A$ and, for $x \in A$, let $d(x, B) = \inf\{ d(x, b) \mid b \in B \}$. Prove that for all $x, y \in A$, $|d(x, B) - d(y, B)| \leq d(x, y)$. De duce that for any $\varepsilon > 0$, the set $\{ x \in A \mid 0 < d(x, B) < \varepsilon \}$ is open, stating clearly any result you use.

5.5 Final Examination I

1.1 Define the Cantor K to be the set of real numbers of the form $\sum_{n=1}^{\infty} \frac{a_n}{3^n}$, where each $a_n \in \{0, 2\}$. Is K countable or uncountable? Prove your answer.[6]

1.2 Prove directly from the definition that if the sequence (x_n) of real numbers converges, then $\lim_{n\to\infty} x_n^2 = \left(\lim_{n\to\infty} x_n\right)^2$. Show that (x_n^2) may be convergent even if (x_n) is not.

1.3 Prove from the definition that the intersection of finitely many open sets is open. Show that the intersection of infinitely many open sets may not

[6] math360exam1sol.pdf

be open.

1.4 If $x_n = \cos\left(\dfrac{(3n^3+n+2)\pi}{6}\right)$, determine $\limsup_{n\to\infty} x_n$. Prove your answer is correct.

1.5 (a) Prove directly from the definition that the set $S = \{\ (x,y) \in \mathbb{R}^2 \mid y > 0\ \}$ is open.

(b) Prove that the definition that S is not closed.

1.6 Prove that the sequence (x_n), defined inductively by $x_1 = 1, \quad x_{n+1} = \sqrt{2+x_n}$, is convergent. What is its limit?

5.6 Final Examination II

2.1 (a) If A is connected, prove that the closure $\mathrm{cl}(A)$ is connected. [7]

(b) Show that if A is path-connected, $\mathrm{cl}(A)$ may not be path-connected.

2.2 (a) Using sequences, prove that if f is continuous, then $f[K]$ is compact whenever K is compact, and $f^{-1}[C]$ is closed whenever C is closed.

(b) Show that the inverse image of a compact set may not be compact, and that the image of a closed set may not be closed.

2.3 Define the distance between two sets A and B in $\mathbb{R}^2$ to be $D(A,B) = \inf\{\ \|x-y\| \mid x \in A, y \in B\ \}$. Suppose that A and B are disjoint. Show that

(a) If A and B are compact, then $D(A,B) > 0$.

(b) If A and B are just closed, then $D(A,B)$ may be equal to zero. What happens if A is compact and B is closed? Justify your answer.

[7] math360exam2sol.pdf

2.4 (a) Let f be a bounded function on $[a,b]$. Prove that if there is a sequence $\mathcal{P}$ of partitions of $[a,b]$ such that $I = \underset{n\to\infty}{} U(f,\mathcal{P}_n) = \underset{n\to\infty}{} L(f,\mathcal{P}_n)$, then f is integrable and $\displaystyle\int_a^b f(x)dx = I$.

(b) Suppose that

$$f(x) = \begin{cases} 0 & \text{if } -1 \le x < 0; \\ 1 & \text{if } 0 \le x \le 1. \end{cases}$$

Use the result above to prove directly that f is integrable and to compute $\displaystyle\int_{-1}^{1} f(x)dx$.

2.5 Show that if f'' exists and is continuous on $[0,\infty)$ then $f(x) = f(0) + f'(0)x + \displaystyle\int_0^x (x-t)f''(t)dt$, for all $x \ge 0$.

6 Review Exercises

6.1 Interval

1. If $I = [a,b]$ and $J = [c,d]$ are closed intervals in $\mathbb{R}$. Show that $I \subset J$ if and only if $c \le a$ and $b \le d$. What happens if I and J are open intervals?

2. If $S \subset \mathbb{R}$ is non-empty, show that S is bounded if and only if there is some closed interval I such that $S \subset I$.

3. Let S be a non-empty bounded subset of $\mathbb{R}$, show that

   (a) $S \subset [\ \inf S,\ \sup S\ ]$.

   (b) If J is a closed interval such that $S \subset J$, then $[\ \inf S,\ \sup S\ ] \subset J$.

4. Let $I_n = [a_n, b_n]$ $(n \ge 1)$ be a collection of closed intervals, prove that $I_1 \supset I_2 \supset \cdots$ (i.e. they are nested intervals) if and only if $a_1 \le a_2 \le \cdots$ and $b_1 \ge b_2 \ge \cdots$.

5. Let $I_n = (0, 1/n)$ for $n = 1, 2, \cdots$. Prove that $\displaystyle\bigcap_{n=1}^{\infty} I_n = \emptyset$.

6. Prove that every point of closed interval $[0,1]$ is a cluster point of open interval $(0,1)$.

7. Show that a finite subset in $\mathbb{R}$ has no cluster points.

8. If $x > 0$ and $0 < \varepsilon < x$, show that there are at most finitely many positive integers n such that $1/n \in (x-\varepsilon, x+\varepsilon)$.

9. Prove that every point of $I = [0,1]$ is a cluster point of $I \cap \mathbb{Q}$ and $I \setminus \mathbb{Q}$ respectively.

10. Suppose that a_k $(k = 1, 2, \cdots, n)$ and b_k $(k = 1, 2, \cdots, m)$ all belong to $\{0, 1, \cdots, 8, 9\}$ and that $\dfrac{a_1}{10^1} + \dfrac{a_2}{10^2} + \cdots + \dfrac{a_n}{10^n} = \dfrac{b_1}{10^1} + \dfrac{b_2}{10^2} + \cdots + \dfrac{b_m}{10^m} \neq 0$. Show that (i) $n = m$ and (ii) $a_k = b_k$ $(k = 1, 2, \cdots, n)$.

6.2 Cauchy Criterion

1. Give an example of a bounded sequence that is not a Cauchy sequence.

2. Show directly that the following are Cauchy sequences:
   (a) $x_n = \frac{n+1}{n}$; (b) $y_n = 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}$.

3. Show directly that the following are not Cauchy sequences:
   (a) $x_n = (-1)^n$; (b) $y_n = n + \dfrac{(-1)^n}{n}$.

4. Show directly that if (x_n) and (y_n) are Cauchy sequences, then the sequence $(x_n + y_n)$ and $(x_n \cdot y_n)$ are Cauchy.

5. Let (x_n) be a Cauchy sequence such that x_n is an integer for all $n \in \mathbb{N}$. Show that (x_n) is ultimately constant.

6. Show directly that a bounded monotone increasing sequence is a Cauchy sequence.

7. If $x_1 < x_2$ are arbitrary real numbers and $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$ for all $n > 2$, show that (x_n) is convergent. What is its limit?

8. If $x_1 > 0$ and $x_{n+1} = \frac{1}{2+x_n}$ for all $n > 1$, show that (x_n) is contractive sequence. Find the limit.

9. The polynomial equation $x^3 - 5x + 1 = 0$ has a root r with $0 < r < 1$. Use an appropriate contractive sequence to calculate r with 10^{-4}.

6.3 Limits of Functions

1. Determine a condition on the range of $|x - 1|$ that will assure that

   (a) $|x^2 - 1| < 1/2$;

   (b) $|x^2 - 1| < 1/10^3$;

   (c) $|x^2 - 1| < 1/n$, for a given natural number $n \in \mathbb{N}$;

   (d) $|x^3 - 1| < 1/n$, for a given natural number $n \in \mathbb{N}$;

2. Let c be a limit point of $A \subset \mathbb{R}$ and let $f : A \to \mathbb{R}$. Prove that $\lim\limits_{x \to c} f(x) = L$ if and only if $\underset{x \to c}{|} f(x) - L| = 0$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ and let $c \in \mathbb{R}$. Show that $\lim\limits_{x \to c} f(x) = L$ if and only if $\lim\limits_{x \to 0} f(x + c) = L$.

4. Let $f : \mathbb{R} \to \mathbb{R}$, $I \subset \mathbb{R}$ be an open interval and $c \in I$. If $f_1 = f|_I$ be the restriction of f onto I. Show that (i) f_1 has a limit at c if and only if f has a limit at c, and (ii) their limits are the same.

5. Let $f : \mathbb{R} \to \mathbb{R}$, $J \subset \mathbb{R}$ be a a closed interval, and $c \in J$. If $f_2 = f|_J$, show that if f has a limit at c, then f_2 has a limit at c. Show that the converse does not hold.

6. Let $I = (0, a)$ be an open interval with $a > 0$, and $g(x) = x^2$ for all $x \in I$.
   (i) For any x, $c \in I$, show that $|g(x) - c2| \leq 2a|x - c|$.
   (ii) Use the inequality above to prove that $\underset{x \to c}{} f(x) = c^2$, for any $c \in I$.

7. Let $I \subset \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ be a function and $c \in I$. Suppose that there exists numbers K and L such that $|f(x) - L| \leq K|x - c|$ for all $x \in I$. Show that $\lim_{x \to c} f(x) = L$.

8. Show that $\lim_{x \to c} x^3 = c^3$.

9. Show that $\lim_{x \to c} \sqrt{x} = \sqrt{c}$ for any $c > 0$.

10. Use both $\varepsilon - \delta$ and the sequential formulations of the notion of a limit to establish the following:

    (a) $\lim_{x \to 2} \dfrac{1}{1 - x} = -1$ for $x > 1$.

    (b) $\lim_{x \to 1} \dfrac{x}{1 + x} = \dfrac{1}{2}$ for $x > 0$.

    (c) $\lim_{x \to 0} \dfrac{x^2}{|x|} = 0$ for $x \neq 0$.

    (d) $\lim_{x \to 1} \dfrac{x^2 - x + 1}{x + 1} = \dfrac{1}{2}$ for $x > 0$.

11. Show that the following limits *do not exist* in $\mathbb{R}$:

    (a) $\lim_{x \to 0} \dfrac{1}{x^2}$ for $x > 0$.

    (b) $\lim_{x \to 0} \dfrac{1}{\sqrt{x}}$ for $x > 0$.

    (c) $\lim_{x \to 0} (x + \operatorname{sgn}(x)\ )$.

    (d) $\lim_{x \to 0} \sin \dfrac{1}{x^2}$ for $x \neq 0$.

12. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ has limit L at 0, and let $a > 0$. If $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = f(ax)$ for all $x \in \mathbb{R}$, show that $\lim_{x \to 0} g(x) = L$.

13. Let c be a limit point of A ($\subset \mathbb{R}$), and let $f : A \to \mathbb{R}$ be such that $\lim_{x \to c} f(x)^2 = L$. Show that if $L = 0$, then $\lim_{x \to 0} f(x) = 0$. Show by example that if $L \neq 0$, then f may not have a limit at c.

6.4 Limits

1. Determine the following limits:

    (a) $\lim_{x \to 1} (x + 1)(2x + 3)$.

    (b) $\lim_{x \to 1} \dfrac{x^2 + 2}{x^2 - 1}$.

    (c) $\lim_{x \to 2} \left(\dfrac{1}{x + 1} - \dfrac{1}{2x} \right)$.

    (d) $\lim_{x \to 0} \dfrac{|x + 1|}{x^2 + 2}$.

2. Determine the following limits:

    (a) $\lim_{x \to 2} \sqrt{\dfrac{2x + 1}{x + 3}}$.

    (b) $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2}$.

    (c) $\lim_{x \to 2} \dfrac{(x + 1)^2 - 1}{x}$.

    (d) $\lim_{x \to 0} \dfrac{\sqrt{x} - 1|}{x - 1}$.

3. Find $\lim_{x \to 0} \dfrac{\sqrt{1 + 2x} - \sqrt{1 + 3x}}{x + 2x^2}$, where $x > 0$.

4. Prove that $\lim_{x \to 0} \cos(\dfrac{1}{x})$ does not exist but that $\lim_{x \to 0} x \cos(\dfrac{1}{x}) = 0$.

5. Let $f, g : A \to \mathbb{R}$ be two functions, and c is a limit point of A. Suppose that f is bounded on a neighborhood of c and that $\lim_{x \to c} g(x) = 0$. Prove that $\lim_{x \to c} (f(x) \cdot g(x)) = 0$.

6. Let $n \geq 3$ be a natural number. Show that $-x^2 \leq x^n \leq x^2$ for all $-1 < x < 1$. Then use the fact that $\lim_{x \to 0} x^2 = 0$ to show that $\lim_{x \to 0} x^n = 0$.

7. Let $f, g : A \to \mathbb{R}$ and c is a limit point of A.

    (a) Show that if both $\lim_{x \to c} f(x)$ and $\lim_{x \to c} (f(x) + g(x))$ exist, then $\lim_{x \to c} g(x)$ exists.

(b) If both $\lim_{x \to c} f(x)$ and $\lim_{x \to c} (f(x) \cdot g(x))$ exist, does it follow that $\lim_{x \to c} g(x)$ exists?

8. Give examples of functions f and g defined on the same domain such that f and g do not have limits at a point c, but such that both $f \cdot g$ and $f + g$ have limits at c.

9. Determine whether the following limits exist in $\mathbb{R}$:

   (a) $\lim_{x \to 0} \sin(\frac{1}{x^2})$ for $x \neq 0$.

   (b) $\lim_{x \to 0} x \sin(\frac{1}{\sqrt{x^2}})$ for $x \neq 0$.

   (c) $\lim_{x \to 0} \text{sgn}(\ \sin(1/x)\ )$.

   (d) $\lim_{x \to 0} \sqrt{x} \sin \frac{1}{x^2}$ for $x \neq 0$.

10. Give examples of functions f and g such that f and g do not have limits at a point c, but such that both $f + g$ and $f \cdot g$ have limits at c.

11. Determine whether the following limits exist in $\mathbb{R}$:

    (a) $\lim_{x \to 0} \sin(\frac{1}{x^2})$.

    (b) $\lim_{x \to 0} x \sin(\frac{1}{x^2})$.

    (c) $\lim_{x \to 0} \text{sgn}(\ \sin(\frac{1}{x})\ )$.

    (d) $\lim_{x \to 0} \sqrt{x} \sin(\frac{1}{x^2})$.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathbb{R}$. Assume that $\lim_{x \to 0} f(x) = L$. Prove that $L = 0$, and then prove that f has a limit at every point $c \in \mathbb{R}$.
    Hint: First note that $f(2x) = f(x) + f(x) = 2f(x)$ for all $x \in \mathbb{R}$. Also note that $f(x) = f(x - c) + f(c)$. for $x, c \in \mathbb{R}$.

6.5 True and False

Here are some common attempts to define the notion of a convergent sequence. Study them and see why they are incorrect. In order to reinforce your understanding, create a sequence in each case illustrating what is wrong with the definition. (I provided one such example in the first case.)

1. A set A is countable if and only if it is finite.

2. A set A is countable if and only if there exist a surjection from A onto $\mathbb{N}$.

3. $A \times B = \{\ a \cdot b \mid a \in A \text{ and } b \in B\ \}$.

4. $f : X \to \mathbb{R}$ is uniformly continuous if and only if f is continuous.

5. A sequence converges to x if there exists an N such that for all $n > N$ and all $\varepsilon > 0$, $|s_n - x| < \varepsilon$.
   This definition is too strong, since with this definition the sequence $\{\ 1/n \mid n \geq 1\ \}$ does not converge to zero. There does not exist any N which works for all $\varepsilon > 0$. Indeed, given any $N > 0$, we can choose $\varepsilon = \frac{1}{N+2}$, and then $|s_{n+1} - 0| = \frac{1}{n+1} > \varepsilon$.

6. A sequence (x_n) converges to x if for all $\varepsilon > 0$ there exists an $n > N$ such that $|s_n - x| < \varepsilon$.

7. A sequence converges to x if there exists an n_0 such that for all $n > n_0$, there exists an $\varepsilon > 0$ such that $|s_n - x| < \varepsilon$.

8. A sequence converges to x if there exists an N and an ε such that for all $n > N$, $|s_n - x| < \varepsilon$.

9. A sequence converges to x if $|s_n - x| < \varepsilon$ for all n, where $\varepsilon > 0$.

6.6 Important points for review

1. Prove that the series $\sum \frac{1}{n}$ diverges.

2. Prove that the series $\sum \frac{1}{n^\alpha}$ converges if $\alpha > 1$.

3. Properties of the set $\mathbb{N}$ of natural numbers.

   (a) smallest inductive set.

   (b) principle mathematical induction.

   (c) well-ordering principle, existence of minimal element for any non-empty subset of $\mathbb{N}$.

4. In order to prove that a subset S of $\mathbb{R}$ has a finite supremum, one needs to establish the following:

   (a) S is non-empty

   (b) S has an upper bound, or S is bounded above.

   (c) Apply the Supremum principle to claim the existence. Usually, it is very difficult to find out the supremum.