HARDER INDUCTION – WORKSHEET

COURSE/LEVEL

NSW Secondary High School Year 11 Preliminary Mathematics Extension. Syllabus reference: 7.4.

1. Prove the following by induction, where n is any positive integer:

(a) $1^2 + 3^2 + 5^2 + \ldots (2n+1)^2 = \dfrac{(n+1)(2n+1)(2n+3)}{6}$.

(b) $\dfrac{1}{1.2.3} + \dfrac{3}{2.3.4} + \dfrac{5}{3.4.5} + \dfrac{7}{4.5.6} + \cdots + \dfrac{(2n-1)}{n(n+1)(n+2)} = \dfrac{3}{4} + \dfrac{1}{2(n+1)} - \dfrac{5}{2(n+2)}$.

(c) $1\times 1! + 2\times 2! + 3\times 3! + n\times n! + = (n+1)! - 1$.

(d) $\dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \cdots + \dfrac{n}{(n+1)!} = 1 - \dfrac{1}{(n+1)!}$.

(e) $\displaystyle\sum_{i=1}^{n} i(n-i+1) = \frac{n(n+1)(n+2)}{6}$.

(f) $1 + 2x + 3x^2 + \ldots + nx^{n-1} = \dfrac{1-x^n}{(1-x)^2} - \dfrac{nx^n}{1-x}$.

(g) $\displaystyle\sum_{i=1}^{n} \left(i^3 + 3i^5\right) = 4\left(\sum_{i=1}^{n} i\right)^3$.

(h) $8 + 88 + 888 + \ldots + \underbrace{888\ldots8}_{n \text{ digits}} = \frac{8}{81}\left(10^{n+1} - 9n - 10\right)$.

(i) $\cos\theta + \cos 2\theta + \ldots + \cos n\theta = \dfrac{\sin\frac{1}{2}(2n+1)\theta}{2\sin\frac{1}{2}\theta} - \dfrac{1}{2}$.

(j) $\sin\theta + \sin 2\theta + \ldots + \sin n\theta = \dfrac{\sin\frac{1}{2}n\theta \sin\frac{1}{2}(n+1)\theta}{\sin\frac{1}{2}\theta}$.

Hint: You may use the fact that $2\cos x \sin y = \sin(x+y) - \sin(x-y)$.

(k) Prove the Binomial Theorem:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \quad \text{where} \quad \binom{n}{k} = \frac{n!}{(n-k)!\,k!}.$$

Hint: You may use the fact that $\dbinom{n}{k-1} + \dbinom{n}{k} = \dbinom{n+1}{k}$.

(j) If $y=\sin(ax)$, then $\dfrac{d^n y}{dx^n}=a^n\sin\left(ax+\dfrac{n\pi}{2}\right)$.

(k) If $y=\ln(1+x)$, then $\dfrac{d^n y}{dx^n}=\dfrac{(-1)^{n-1}(n-1)!}{(1+x)^n}$.

(l) If $y=\dfrac{1}{1-x}$, then $\dfrac{d^n y}{dx^n}=\dfrac{n!}{(1-x)^{n+1}}$.

(m) If $y=\dfrac{1}{1-x^2}$, then $\dfrac{d^n y}{dx^n}=\dfrac{n!}{2}\left(\dfrac{1}{(1-x)^{n+1}}+\dfrac{(-1)^n}{(1+x)^{n+1}}\right)$.

2. Prove the following results for all **odd integers** $n\ge 1$.

(a) 3^n+7^n is divisible by 10 .

(b) 7^n+11^n is divisible by 9.

3. Prove that each of the following expressions are divisible by 9 if *n* is any positive integer.

(a) 4^n+6n-1.

(b) $5^{2n}+3n-1$.

(c) $(3n+1)7^n-1$.

(d) $n^3+(n+1)^3+(n+2)^3$.

4. Prove that $x^{2n}-y^{2n}$ is divisible by $x+y$ where *n* is any positive integer.

5. Prove the following for all integer values of *n* not less than 1.

(a) $\displaystyle\sum_{k=1}^{2^n}\frac{1}{k}\ \ge\ \frac{n+2}{2}$.

(b) $\displaystyle\sum_{k=1}^{n}\frac{1}{k^2}\ \le\ 2-\frac{1}{n}$. (HSC 1987)

(c) $\displaystyle\sum_{k=1}^{2^n-1}\frac{1}{k^2}\ \le\ 2-\frac{1}{2^{n-1}}$

(d) $\displaystyle\sum_{k=1}^{2^n-1}\frac{1}{k^r}\ \le\ \frac{1-\left(\frac{1}{2}\right)^{(r-1)n}}{1-\left(\frac{1}{2}\right)^{r-1}}$

(Note that (c) is a special case of the inequality in (d) with $r=2$.)

6. (HSC 1985)

(a) Show that for $k\ge 0$, $2k+3>2\sqrt{(k+1)(k+2)}$.

(b) Hence prove that for $n\ge 1$,

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}>2\left(\sqrt{n+1}-1\right).$$

7. Prove Bernoulli's inequality: $1+nx\le(1+x)^n$ where $x>-1$ (and *n* is any positive integer.)