COURSE/LEVEL
NSW Secondary High School Year 12 HSC Extension 2 Mathematics.

HSC TRIAL EXAMINATION

MATHEMATICS

Extension 2

Time allowed - Three hours

DIRECTIONS

- Attempt ALL questions
- EACH question is out of 15 marks
- All necessary working should be shown. Marks may be deducted for careless or poorly arranged work
- Start each question on a new page

QUESTION 1.

(a) *(3 marks)*

Find $\int \sin^3 x \, dx$

(b) *(4 marks)*

Using the substitution $t = \tan\left(\frac{\theta}{2}\right)$, or otherwise, show that

$$\int_0^{\pi/2} \frac{1}{1+\sin\theta} \, d\theta = 1.$$

(c) *(4 marks)*

Evaluate $\int_0^1 \tan^{-1} x \, dx$

(d) *(4 marks)*

(i) Express

$$\frac{3-x}{\left(1+2x^2\right)\left(1+6x\right)}$$

in partial fractions.

(ii) Show that

$$\int_0^2 \frac{3-x}{\left(1+2x^2\right)\left(1+6x\right)} \, dx = \frac{1}{2} \ln\left(\frac{13}{3}\right).$$

QUESTION 2.

(a) *(3 marks)*

Given that $(2+3i)\,p-q=1+2i$, find p and q if

(i) p and q are real

(ii) p and q are complex conjugate numbers

(b) *(3 marks)*

If $z=\cos\theta+i\sin\theta$, show that

$$\frac{1}{1+z}=\frac{1}{2}\left(1-i\tan\frac{\theta}{2}\right)$$

(c) *(4 marks)*

(i) On an Argand diagram, shade in the region for which

$$0\le|z|\le 2 \quad \text{and} \quad 1\le \operatorname{Im} z\le 2$$

(ii) Write down the complex number with largest argument that satisfies the inequalities of (i). Express your answer in the form $a+ib$.

(d) *(5 marks)*

(i) Find the two square roots of $5-12i$ in the form $x+iy$ where x and y are real.

(ii) Show the points P and Q representing the square roots on an Argand diagram. Find the complex numbers represented by points R_1, R_2 such that the triangles PQR_1 and PQR_2 are equilateral.

QUESTION 3.

(a) *(5 marks)*

The rate of change, with respect to x, of the gradient of a curve is constant and the curve passes through the points $(1,2)$ and $(-3,0)$, the gradient at the former point being $-\frac{1}{2}$. Find the equation of the curve and sketch the curve.

(b) *(10 marks)*

For the ellipse $x^2 + 4y^2 = 100$,

(i) Write down the eccentricity, the co-ordinates of the foci and the equations of the directrices.

(ii) Sketch a graph of the ellipse showing the above features.

(iii) Find the equation of the tangent and normal to the ellipse at the point $P(8,3)$.

(iv) If the normal at P meets the major axis at G and the perpendicular from the centre 0 to the tangent at P meets that tangent at K, prove that $PG.OK$ is equal to the square of the minor semi-axis.

QUESTION 4.

(a) *(6 marks)*

(i) If $P(x) = x^3 - 9x^2 + 24x + c$ for some real number c, find the values of x for which $P'(x) = 0$. Hence find the two values of c for which the equation $P(x) = 0$ has a repeated root.

(ii) Sketch the graphs of $y = P(x)$ for these values of c. Hence write down the values of c for which the equation $P(x) = 0$ has three distinct real roots.

(b) *(6 marks)*

Let $f(x) = x - 2 + \dfrac{3}{x+2}$.

(i) Find the points at which $f(x) = 0$.

(ii) Find the turning points of $f(x)$, if any, and identify them.

(iii) Find the asymptotes.

(iv) Sketch the curve, marking all the features you have found in parts (i) - (iii) above.

(c) *(3 marks)*

The polynomial $x^3 + x^2 + 3x - 2 = 0$ has roots α, β and γ. Find the equation with roots $\alpha^2\beta\gamma$, $\alpha\beta^2\gamma$ and $\alpha\beta\gamma^2$.

QUESTION 5. *(15 marks)*

A particle of mass m is projected vertically upwards under gravity in a medium which exerts a resisting force of magnitude $mg(v/k)^2$, where v is the speed of the particle and k is a constant.

(i) For the upward motion of the particle, draw a diagram showing the forces acting on the particle and write down the equation of motion.

(ii) If U is the speed of projection, show that the greatest height of the particle above the point of projection is

$$\frac{k^2}{2g} \ln\left(\frac{k^2 + U^2}{k^2}\right).$$

(iii) Repeat part (i) for the downward motion of the particle and hence write down the particle's terminal velocity.

(iv) If V is the speed of the particle on returning to the point of projection, show that

$$\frac{1}{V^2} - \frac{1}{U^2} = \frac{1}{k^2}.$$

QUESTION 6.

(a) *(3 marks)*

Let $\min(a,b)$ denote the minimum of the numbers *a* and *b*. Sketch the function $y = \min(2,x)$ over the interval $0 \le x \le 3$ and evaluate $\int_0^3 \min(2,x)\,dx$.

(b) *(3 marks)*

Find the area enclosed between the curves $y = x^3$ and $y^3 = 16x$.

(c) *(9 marks)*

(i) Sketch the curves $y = \tan x$ and $y = 2\cos\left(x + \frac{\pi}{12}\right)$ between $x = 0$ and $x = \frac{\pi}{2}$

(ii) Verify that $x = \frac{\pi}{4}$ is a solution of the equation $\tan x - 2\cos\left(x + \frac{\pi}{12}\right) = 0$.

(iii) Find the area enclosed by these curves and the *y*-axis.

(iv) If this area is rotated through one revolution about the *x*-axis, find the volume of the solid formed.

QUESTION 7.

(a) *(7 marks)*

Two circles intersect at *A* and *B*. The tangents from a point on *BA* produced meet the circles at *P* and *Q*.

If *P*, *A* and *Q* are collinear,

(i) Draw a diagram showing this information.

(ii) Prove that $\Delta TAP \parallel\!| \Delta TBP$ and $\Delta TAQ \parallel\!| \Delta TBQ$.

(iii) Prove that *T*, *Q*, *B*, *P* are concyclic.

(iv) Prove that $TP = TQ$.

(b) (*8 marks*)

For a given integer $n \geq 1$, let the positive integers $c_0, c_1, \ldots, c_n$ be defined by the equation, valid for all (real and) complex numbers *z*:

$$(1+z)^n = c_0 + c_1 z + \ldots + c_n z^n .$$

(You are **not** required to establish this identity.)

Prove that

(i) $c_0 = 1$,

(ii) $c_0 - c_1 + c_2 - c_3 + \ldots + (-1)^n c_n = 0$,

(iii) if *n* is odd then $c_1 + c_3 + \ldots + c_{n-2} + c_n = 2^{n-1}$,

(iv) if n is divisible by 4 then $c_0 - c_2 + c_4 - \ldots - c_{n-2} + c_n = (-1)^{n/4} 2^{n/2}$.

QUESTION 8.

(a) (2 marks)

If the functions $f(x)$ and $g(x)$ are such that $f(x) > g(x) \geq 0$ for $a \leq x \leq b$, by using a sketch (or otherwise) explain why $\int_a^b f(x)dx > \int_a^b g(x)dx$.

(b) (13 marks)

Let

$$u_n = \int_0^1 \left(1 - t^2\right)^{(n-1)/2} dt$$

where *n* is a non-negative integer.

(i) Using integration by parts, or otherwise, show that $n u_n = (n-1)u_{n-2}$ if $n \geq 2$.

(ii) Let $v_n = n\, u_n u_{n-1}$, $n \geq 1$. Show that $v_n = \frac{1}{2}\pi$, for all values of $n \geq 1$.

(iii) Using part (a), or otherwise, show that $0 < u_n < u_{n-1}$. Prove that

$$\sqrt{\frac{\pi}{2n+2}} < u_n < \sqrt{\frac{\pi}{2n}}$$