Section 4.3 - 4.5 - Additional Exponential Models

We will begin by examining how to differentiate exponential and logarithmic functions with bases other than e. To find the derivative of $\log_b(x)$, we use our change of base formula to write $\log_b(x)$ in terms of the natural log and then differentiate. For b^x, we can use the fact that the natural log and e^x are inverse functions to accomplish the job:

Differentiate:

Ex. 1 $y = \log_b(x)$

Solution:

Since $\log_b(x) = \frac{\ln(x)}{\ln(b)}$ by the change of base formula, then

$\frac{d}{dx}[\log_b(x)] = \frac{d}{dx}[\frac{\ln(x)}{\ln(b)}] = \frac{1}{\ln(b)} \bullet \frac{d}{dx}[\ln(x)] = \frac{1}{\ln(b)} \bullet \frac{1}{x} = \frac{1}{x\ln(b)}$.

Ex. 2 $y = b^x$

Solution:

$b^x = e^{\ln(b^x)}$ since e^x and $\ln(x)$ are inverse functions.

$= e^{x\ln(b)}$ by prop. #3 of logs

Thus, $\frac{d}{dx}[b^x] = \frac{d}{dx}[e^{x\ln(b)}] = e^{x\ln(b)} \bullet \frac{d}{dx}[x\ln(b)]$

$= e^{x\ln(b)} \bullet \ln(b)$. But $b^x = e^{x\ln(b)}$, so

$\frac{d}{dx}[b^x] = e^{x\ln(b)} \bullet \ln(b) = b^x\ln(b)$.

Differentiate exponential and logarithmic functions with bases other than e

1) If $y = \log_b(x)$, then $y' = \frac{1}{x\ln(b)}$ and

if $y = \log_b(f(x))$, then $y' = \frac{1}{f(x)\ln(b)} \bullet f'(x) = \frac{f'(x)}{f(x)\ln(b)}$.

2) If $y = b^x$, then $y' = b^x\ln(b)$ and
If $y = b^{f(x)}$, then $y' = b^{f(x)}\ln(b) \bullet f'(x)$

Differentiate:

Ex. 3 a) $f(x) = \log_2(x)$ b) $g(x) = \log_{15}(x)$

c) $h(x) = \log_{1/3}(x^2 - 3x + 4)$

Solution:

a) $f'(x) = \frac{d}{dx}[\log_2(x)] = \frac{1}{x\ln(2)}$.

b) $g'(x) = \frac{d}{dx}[\log_{15}(x)] = \frac{1}{x\ln(15)}$.

c) $h'(x) = \frac{d}{dx}[\log_{1/3}(x^2 - 3x + 4)]$

$= \frac{1}{(x^2 - 3x + 4)\ln(1/3)} \bullet \frac{d}{dx}[x^2 - 3x + 4]$

$= \frac{1}{(x^2 - 3x + 4)\ln(3^{-1})} \bullet [2x - 3] = -\frac{2x - 3}{(x^2 - 3x + 4)\ln(3)}$.

Ex. 4 a) $y = 5^x$ b) $g(x) = (\frac{1}{2})^x$ c) y =

Solution:

a) $y' = \frac{d}{dx}[5^x] = 5^x\ln(5)$

b) $y' = \frac{d}{dx}[(\frac{1}{2})^x] = (\frac{1}{2})^x\ln(\frac{1}{2}) = (\frac{1}{2})^x\ln(2^{-1}) = -(\frac{1}{2})^x\ln(2)$.

c) $y' = \frac{d}{dx}[3^{x^2 - 3x}] = 3^{x^2 - 3x}\ln(3)\frac{d}{dx}[x^2 - 3x]$

$= 3^{x^2 - 3x}\ln(3)(2x - 3) = (2x - 3)3^{x^2 - 3x}\ln(3)$.

Logistic Curves

The graph of the function $Q(t) = \frac{B}{1 + Ae^{-Bkt}}$, where A, B, and k are positive constants and $t \geq 0$, is called a logistic curve. The growth of many types of populations are modeled by this curve when environmental factors gradually impede the growth of the population. This is the model of population growth that always sought. However, there are times when the environmental factors are not enough to impede growth effectively and more drastic action has to be taken to avoid an environmental disaster. Let's explore an example of logistic model that might model the world population growth:

Ex. 5 Suppose that the world population can be modeled by $P(t) = \frac{24}{2 + 6e^{-0.03t}}$ where t in the number of years after 1960 and P(t) is the world population in billion of people. Sketch a graph of P(t) for $0 \leq t \leq 200$.

Solution:

We will follow the eight step process in graph functions:

I) The domain of P(t) is given by the model: [0, 200]. Note that $2 + 6e^{-0.03t} \neq 0$ since $e^{-0.03t} > 0$.

II) Since $P(0) = \frac{24}{2+6e^{-0.03(0)}} = \frac{24}{2+6e^{0}} = \frac{24}{2+6} = 3$, the P - intercept is (0, 3). Setting P(t) = 0 and solving yields:

$$\frac{24}{2+6e^{-0.03t}} = 0 \quad \text{(multiply by } 2 + 6e^{-0.03t}\text{)}$$

$$(2 + 6e^{-0.03t})\bullet \frac{24}{2+6e^{-0.03t}} = 0\bullet(2 + 6e^{-0.03t})$$

24 = 0, no solution

Thus, there are no t-intercepts.

III) $P(t) = \frac{24}{2+6e^{-0.03t}} = 24(2 + 6e^{-0.03t})^{-1}$

Differentiating yields:

$$P'(t) = \frac{d}{dt}[24(2 + 6e^{-0.03t})^{-1}]$$

$$= -24(2 + 6e^{-0.03t})^{-2} \bullet \frac{d}{dt}[2 + 6e^{-0.03t}]$$

$$= -24(2 + 6e^{-0.03t})^{-2} \bullet [6e^{-0.03t}]\bullet \frac{d}{dt}[-0.03t]$$

$$= -24(2 + 6e^{-0.03t})^{-2} \bullet [6e^{-0.03t}]\bullet [-0.03]$$

$$= \frac{4.32e^{-0.03t}}{(2+6e^{-0.03t})^2}.$$

To find P "(t), we will have to use the quotient rule:

$$P''(t) = \frac{d}{dt}\left[\frac{4.32e^{-0.03t}}{(2+6e^{-0.03t})^2}\right]$$

$$= \frac{(2+6e^{-0.03t})^2 \frac{d}{dt}[4.32e^{-0.03t}] - 4.32e^{-0.03t}\frac{d}{dt}[(2+6e^{-0.03t})^2]}{(2+6e^{-0.03t})^4}$$

$$= \frac{(2+6e^{-0.03t})^2[4.32e^{-0.03t}]\bullet\frac{d}{dt}[-0.03t] - 4.32e^{-0.03t}\bullet 2(2+6e^{-0.03t})\frac{d}{dt}[2+6e^{-0.03t}]}{(2+6e^{-0.03t})^4}$$

$$= \frac{(2+6e^{-0.03t})^2[4.32e^{-0.03t}]\bullet[-0.03] - 4.32e^{-0.03t}\bullet 2(2+6e^{-0.03t})[6e^{-0.03t}]\bullet\frac{d}{dt}[-0.03t]}{(2+6e^{-0.03t})^4}$$

$$= \frac{(2+6e^{-0.03t})^2[4.32e^{-0.03t}]\bullet[-0.03] - 4.32e^{-0.03t}\bullet 2(2+6e^{-0.03t})[6e^{-0.03t}]\bullet[-0.03]}{(2+6e^{-0.03t})^4}$$

Multiply 4.32 by – 0.03 in both parts of the top.

$$= \frac{[-0.1296e^{-0.03t}](2+6e^{-0.03t})^2 + 0.1296e^{-0.03t}\bullet 2(2+6e^{-0.03t})[6e^{-0.03t}]}{(2+6e^{-0.03t})^4}$$

Factor out – $0.1296e^{-0.03t}(2 + 6e^{-0.03t})$ out of the numerator:

$$= \frac{[-0.1296e^{-0.03t}](2+6e^{-0.03t})\{(2+6e^{-0.03t}) - 2[6e^{-0.03t}]\}}{(2+6e^{-0.03t})^4}$$

Reduce:

$$= \frac{[-0.1296e^{-0.03t}]\{(2+6e^{-0.03t}) - 2[6e^{-0.03t}]\}}{(2+6e^{-0.03t})^3}$$

$$= \frac{[-0.1296e^{-0.03t}]\{2+6e^{-0.03t} - 12e^{-0.03t}\}}{(2+6e^{-0.03t})^3}$$

$$= \frac{[-0.1296e^{-0.03t}]\{2-6e^{-0.03t}\}}{(2+6e^{-0.03t})^3}.$$

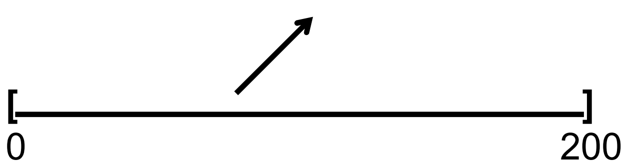
IV) There are no vertical asymptotes, but since $6e^{-0.03t} \to 0$ as $t \to \infty$, then $\lim_{t\to\infty} \frac{24}{2+6e^{-0.03t}} = \frac{24}{2+0}$ = 12. Thus, there is a horizontal asymptote of y = 12.

V) P '(t) is defined for all real numbers. Setting P '(t) = 0 and solving yields:

$\frac{4.32e^{-0.03t}}{(2+6e^{-0.03t})^2} = 0$ (multiply by $(2 + 6e^{-0.03t})^2$)

$(2 + 6e^{-0.03t})^2 \bullet \frac{4.32e^{-0.03t}}{(2+6e^{-0.03t})^2} = 0 \bullet (2 + 6e^{-0.03t})^2$

$4.32e^{-0.03t} = 0$ which has no solution since $4.32e^{-0.03t} > 0$. We need to pick a value in [0, 200] to determine if P is increasing or decreasing:



Pick 100

$$P'(100) = \frac{4.32e^{-0.03(100)}}{(2+6e^{-0.03(100)})^2} = \frac{4.32e^{-3}}{(2+6e^{-3})^2}$$

$$\approx \frac{0.2150801353}{5.284124719} \approx 0.0407.$$

Thus, P is increasing on [0, 200]

VI) N/A since there are no critical values

VII) P "(t) is defined for all real numbers. Setting P "(t) = 0 and solving yields:

$$\frac{[-0.1296e^{-0.03t}]\{2-6e^{-0.03t}\}}{(2+6e^{-0.03t})^3} = 0$$

(multiply by $(2 + 6e^{-0.03t})^3$)

$$(2 + 6e^{-0.03t})^3 \bullet \frac{[-0.1296e^{-0.03t}]\{2-6e^{-0.03t}\}}{(2+6e^{-0.03t})^3} = 0 \bullet (2 + 6e^{-0.03t})^3$$

$$[-0.1296e^{-0.03t}]\{2 - 6e^{-0.03t}\} = 0$$

$-0.1296e^{-0.03t} = 0$ or $2 - 6e^{-0.03t} = 0$

But $-0.1296e^{-0.03t} < 0$ for all t so the only solution comes from:

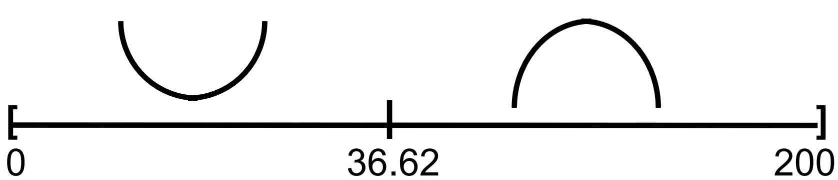
$$2 - 6e^{-0.03t} = 0$$
$$2 = 6e^{-0.03t}$$
$$\frac{1}{3} = e^{-0.03t}$$

Take the natural log of both sides:

$$\ln\left(\frac{1}{3}\right) = \ln\left(e^{-0.03t}\right)$$
$$\ln\left(\frac{1}{3}\right) = -0.03t$$
$$t = \frac{\ln(1/3)}{-0.03} \approx 36.62040962 \approx 36.62.$$

$$P\left(\frac{\ln(1/3)}{-0.03}\right) = \frac{24}{2+6e^{-0.03\left(\frac{\ln(1/3)}{-0.03}\right)}} = \frac{24}{2+6e^{\ln(1/3)}}$$

$= \frac{24}{2+6e^{\ln(1/3)}} = \frac{24}{2+6(\frac{1}{3})} = \frac{24}{2+2} = 6$. Thus, (36.62, 6) is a possible inflection point. Marking 36.62 on the number line, we can determine the concavity of P:

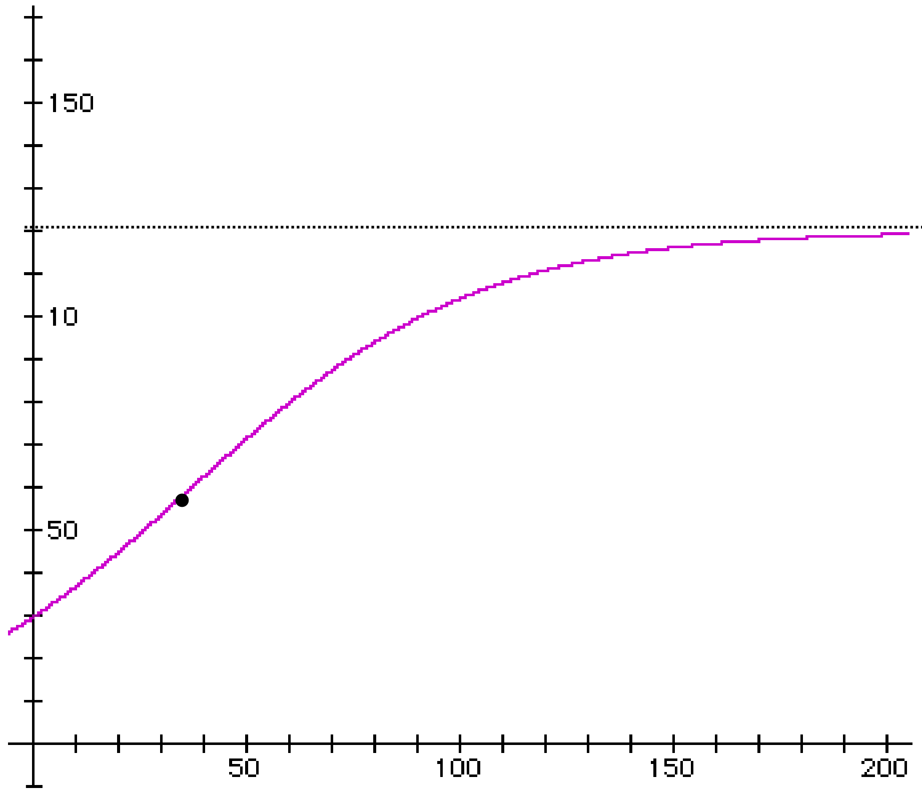


Pick x = 10 | Pick x = 100

P "(10) = 0.000877 | P "(100) = – 0.000904

Thus, P is concave up on [0, 36.62) and concave down on (36.62, 200].

VIII) Now, we can sketch the graph.



A logistics curve will always have this "S" shape. In this example, the initial value of the population was the y-intercept, (0, 3). This says that the world population was 3 billion people in 1960. According to this model, in the long run, the world population will level off at 12 billion people (the horizontal asymptote).

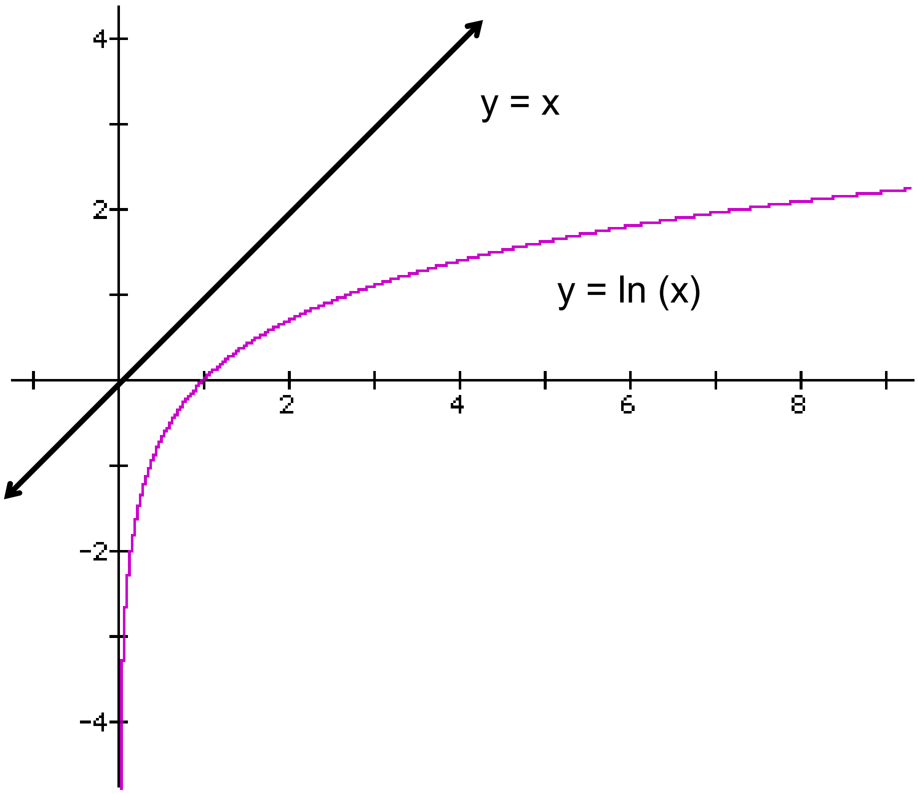
Ex. 6 Sketch the graph of f(x) = x – ln (x).

<u>Solution:</u>

Again, we will use the eight step process:

I) Because of ln (x), the domain of f is (0, ∞).

II) Since f(0) is undefined, then there are no y-intercepts. Setting f(x) = 0 and solving yields:
x – ln (x) = 0
x = ln (x), but x = ln (x) can never be equal. To see why, let's graph y = ln (x) and y = x on the same coordinate axis:



Thus, there are no intercepts.

III) $f'(x) = \frac{d}{dx}[x - \ln(x)] = 1 - \frac{1}{x}$.

$f''(x) = \frac{d}{dx}[1 - \frac{1}{x}] = \frac{d}{dx}[1 - x^{-1}] = x^{-2} = \frac{1}{x^2}$.

IV) Since $\lim_{x \to 0^+} x - \ln(x) = 0 - (-\infty) = \infty$, there is a vertical asymptote of x = 0. Since $\lim_{x \to \infty} x - \ln(x) = \infty$, there is no horizontal asymptote.

V) f '(x) is undefined at x = 0 but x = 0 is not in the domain of f, so it is not a critical value. Setting f '(x) = 0 and solving yields:

$1 - \frac{1}{x} = 0$ (multiply by x)

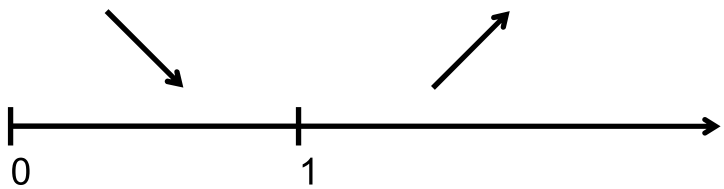
$x \cdot 1 - x \cdot \frac{1}{x} = x \cdot 0$

$x - 1 = 0$

$x = 1$ Evaluating f(1) yields:

$f(1) = 1 - \ln(1) = 1$. Thus, (1, 1) is a critical point.

Marking x = 1 on the number line, we can find where f in increasing and decreasing:



Pick 0.5	Pick 2
f '(0.5)	f '(2)
$= 1 - \frac{1}{0.5}$	$= 1 - \frac{1}{2}$
$= 1 - 2 = -1$	$= \frac{1}{2}$

Thus, f is increasing on $(1, \infty)$ and decreasing on $(0, 1)$.

VI) $f''(1) = \frac{1}{(1)^2} = 1 > 0$, so (1, 1) is a relative minimum.

VII) f "(x) is undefined at x = 0, but x = 0 is not in the domain of f, so x = 0 is not to be considered. Setting f "(x) = 0 and solving yields:

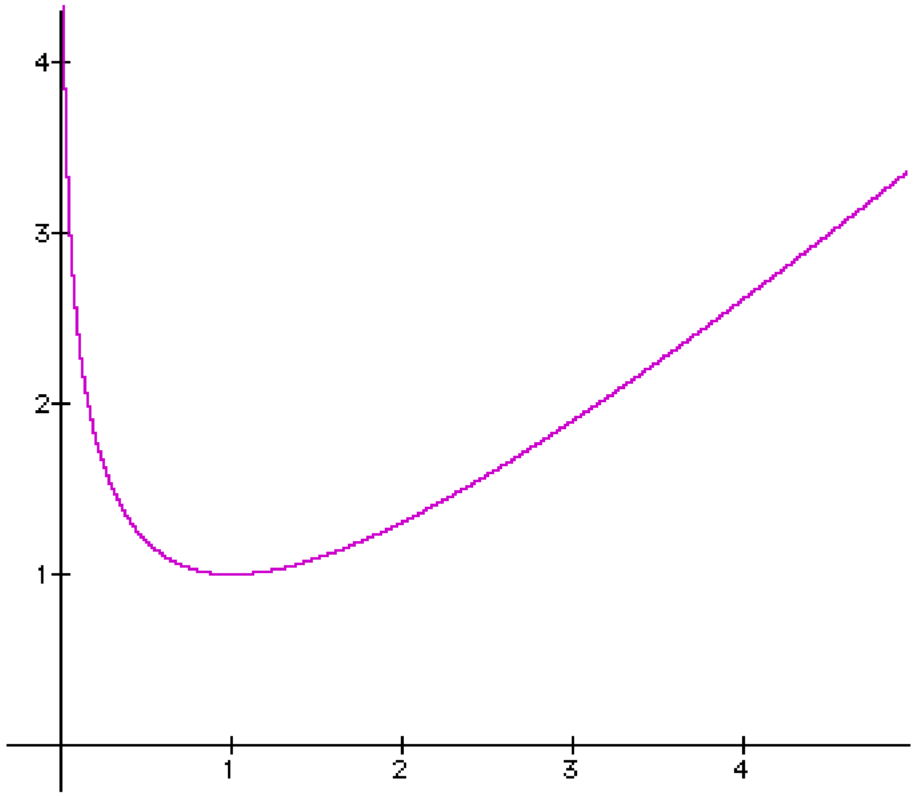
$\frac{1}{x^2} = 0$ (multiply by x^2)

$x^2 \bullet \frac{1}{x^2} = x^2 \bullet 0$

1 = 0, no solution

Thus, there are no inflection points. Since $f''(x) = \frac{1}{x^2} > 0$ for $x > 0$, then f is concave up on $(0, \infty)$.

VIII) Now, we can sketch the graph:



Ex. 6 Suppose a parcel of land will have a market value of V(t) $= 10000e^{\sqrt{t}}$, t years from now. If the prevailing interest rate is 6% compounded continuously, when should the land be sold to maximize its present value.

Solution:

In t years, the present value of the land will be:

$P(t) = V(t)\bullet e^{-0.06t} = 10000e^{\sqrt{t}} \bullet e^{-0.06t} = 10000e^{\sqrt{t}-0.06t}$.

The domain is $[0, \infty)$ because of the context of the problem and P is continuous on $[0, \infty)$. Computing the derivative, we get:

$$P'(t) = \frac{d}{dt}[10000e^{\sqrt{t}-0.06t}]$$

$$= 10000e^{\sqrt{t}-0.06t}\frac{d}{dt}[t^{1/2} - 0.06t]$$

$$= 10000e^{\sqrt{t}-0.06t}\left[\frac{1}{2}t^{-1/2} - 0.06\right]$$

$$= 10000e^{\sqrt{t}-0.06t}\left[\frac{1}{2\sqrt{t}} - 0.06\right]$$

P '(t) is undefined at t = 0 and t = 0 is in the domain of P. Thus, t = 0 is a critical value. Setting P '(t) = 0 and solving yields:

$$10000e^{\sqrt{t}-0.06t}\left[\frac{1}{2\sqrt{t}} - 0.06\right] = 0$$

$$10000e^{\sqrt{t}-0.06t} = 0 \text{ or } \frac{1}{2\sqrt{t}} - 0.06 = 0$$

But $10000e^{\sqrt{t}-0.06t} > 0$, so the only solution comes from: $\frac{1}{2\sqrt{t}} - 0.06 = 0$

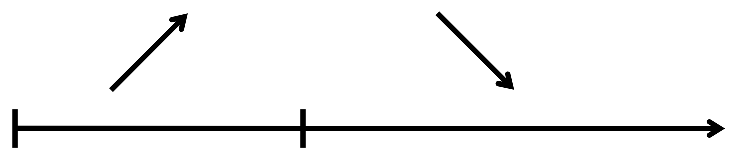
$\frac{1}{2\sqrt{t}} = 0.06$ (multiply by $2\sqrt{t}$)

$1 = 0.12\sqrt{t}$ (square both sides)

$1 = 0.0144t$

$t \approx 69.44$

We have two critical values, t = 0 and t ≈ 69.44. Let's use the first derivative test to determine at which one the maximum value occurs:



0 69.44

Pick 4 Pick 81

P '(4) = P '(81) =

11043.63 – 5459.82

Thus, at t = 0, we have a relative minimum and at t ≈ 69.44, we have a relative maximum. Since the function is increasing to the left of 69.44 and decreasing to the right of 69.44, then the relative maximum is also the absolute maximum. So, the present value is maximized in approximately 69.44 years.