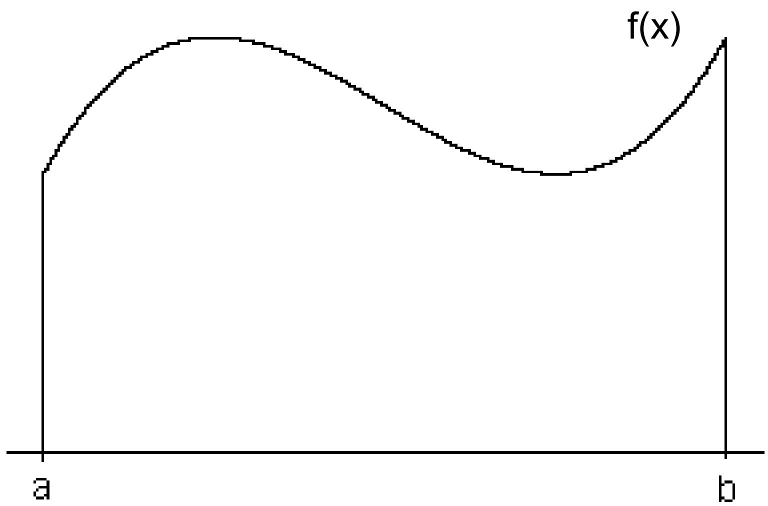
Section 5.3 - Limits of Sums and Accumulations

Recall our discussion from the last section. For any continuous function f(x) where f(x) ≥ 0 for all x in a specified interval [a, b], the $\int$ f(x) dx evaluated at b minus $\int$ f(x) dx evaluated at a will correspond to area under the curve from a to b. We now want to look at the definite integral as the limit of sums Basically, we can start off by approximating the area under a curve by using a series of rectangles under the curve. Then we will find the area of each rectangle and add.

Let f(x) be a continuous, positive function defined on [a, b]



Let n be the number of rectangles that we want to use to approximate the area. The length of each rectangle, Δx, will be the distance between a and b, b – a, divided by the number of rectangles n:

$$\Delta x = \frac{b-a}{n}$$

Let x_1 = a (the value of x that the first rectangle starts,

x_2 = the value of x that the second rectangle starts,

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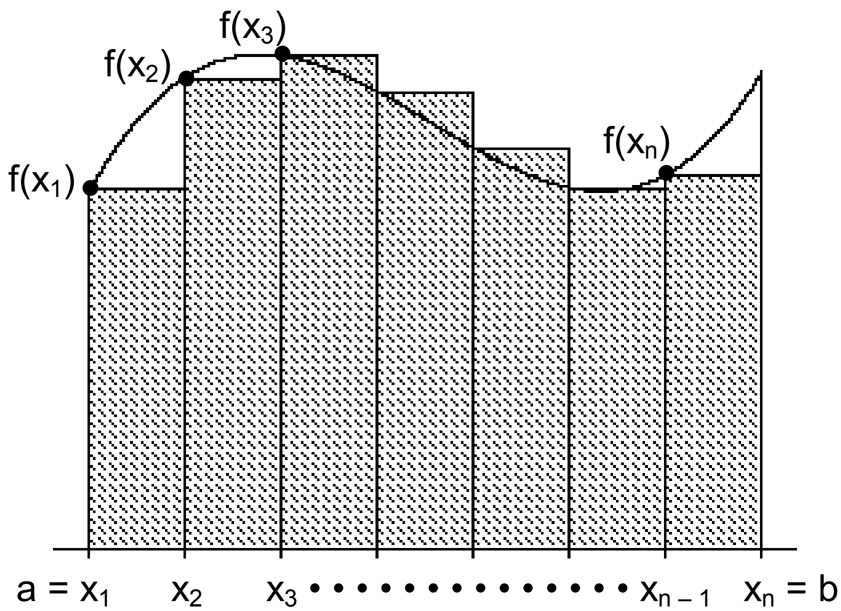
x_n = the value of x that the n^{th} rectangle starts.

Then $f(x_1)$, $f(x_2)$, ..., $f(x_n)$ are the corresponding heights of their respective rectangles.



Thus, the area of rectangles are $f(x_1)\bullet\Delta x$, $f(x_2)\bullet\Delta x$, $f(x_3)\bullet\Delta x$, ..., $f(x_n)\bullet\Delta x$. Hence, the area under the curve is approximately equal to:

$$\sum_{k=1}^{n} f(x_k)\Delta x = f(x_1)\bullet\Delta x + f(x_2)\bullet\Delta x + f(x_3)\bullet\Delta x + ... + f(x_n)\bullet\Delta x$$

To see how this works, let's explore an example.

Ex. 3 Use six rectangles of equal length to approximate the area under the curve $f(x) = x^2 + 1$ on [1, 4].

Solution:

We will begin by constructing six rectangles of length $\Delta x = \frac{b-a}{n} = \frac{4-1}{6} = \frac{3}{6} = \frac{1}{2}$.

Thus, $x_1 = 1$, $x_2 = 1.5$, $x_3 = 2$, $x_4 = 2.5$, $x_5 = 3$, and $x_6 = 3.5$.

The height of the rectangles will then be:

$f(x_1) = f(1) = (1)^2 + 1 = 2$

$f(x_2) = f(1.5) = (1.5)^2 + 1 = 3.25$

$f(x_3) = f(2) = (2)^2 + 1 = 5$

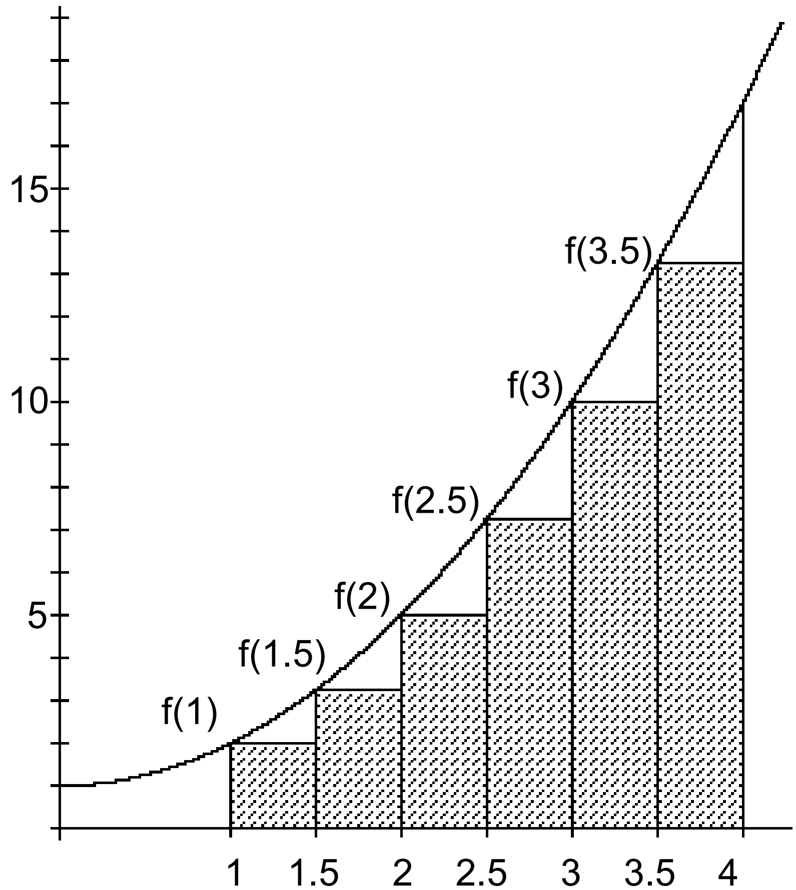
$f(x_4) = f(2.5) = (2.5)^2 + 1 = 7.25$

$f(x_5) = f(3) = (3)^2 + 1 = 10$

$f(x_6) = f(3.5) = (3.5)^2 + 1 = 13.25$

Thus, the areas of the rectangles are:

$f(x_1)\bullet\Delta x$, $f(x_2)\bullet\Delta x$, $f(x_3)\bullet\Delta x$, $f(x_4)\bullet\Delta x$, $f(x_5)\bullet\Delta x$, and $f(x_6)\bullet\Delta x$



The total area of the rectangles is:
$A = f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + f(x_3)\cdot\Delta x + f(x_4)\cdot\Delta x + f(x_5)\cdot\Delta x + f(x_6)\cdot\Delta x$
$= 2(0.5) + 3.25(0.5) + 5(0.5) + 7.25(0.5) + 10(0.5) + 13.25(0.5)$
$= 1 + 1.625 + 2.5 + 3.625 + 5 + 6.625 = 20.375$
Thus, the area under the curve is ≈ 20.375 square units.

The more rectangles we use, the better our approximation will become. If we take the limit as n goes to ∞, Δx will go to zero and we will get the **Definite Integral**:

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty} f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + f(x_3)\cdot\Delta x + \ldots + f(x_n)\cdot\Delta x$$

$$= \int_a^b f(x)\,dx$$

Let's summarize our results and restate the Fundamental Theorem of Calculus.

Area under a curve

If f(x) is continuous and f(x) ≥ 0 on [a, b], then the region under the curve y = f(x) above the interval [a, b] has area of:

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x = \int_a^b f(x)\,dx$$

a is referred to as the lower limit of integration and b is referred as the upper limit of integration.

Fundamental Theorem of Calculus

If f(x) is continuous on [a, b] and F(x) is <u>any</u> antiderivative of f(x), then

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x = \int_a^b f(x)\,dx = F(b) - F(a)$$

The limit of the sums can be viewed as an accumulation of area. This very useful if you are given the rate of some function and are asked to accumulated value. Let us consider the next example.

Ex. 4 For the first two weeks after a product goes on the market, a company estimates that it sales will grow continuously at a rate of $S'(t) = 25e^{0.5t}$, where S'(t) is the sales rate in dollars per day and t is the time in days.

a) Find the total sales for the first six days.

b) Find the total sales from the third day through the eighth day.

c) On what day will the total sales exceed $5,000?

<u>Solution:</u>

a) The accumulated sales represents the total sales for the 1st six days and will correspond to the integral of S'(t) from 0 to 6:

$$\int_0^6 S'(t)\,dt = \int_0^6 25e^{0.5t}\,dt = \frac{25e^{0.5t}}{0.5}\Bigg|_0^6 = 50e^{0.5(6)} - 50e^{0.5(0)}$$

$$= 50e^3 - 50e^0 \approx 1004.27684616 - 50 \approx \$954.28$$

b) The total sales from the third day through the seventh day corresponds to the sales after the second day through the eighth day. So, we need to integrate from 2 to 8:

$$\int_2^8 S'(t)\,dt = \int_2^8 25e^t\,dt = \frac{25e^{0.5t}}{0.5}\Big|_2^8 = 50e^{0.5(8)} - 50e^{0.5(2)}$$

$= 50e^4 - 50e^1 \approx 2729.90750166 - 135.914091423$

$\approx \$2593.99$

c) To find what day the total sales will exceed \$50,000, we will need to integrate from 0 to b and then solve for b:

$$\int_0^b S'(t)\,dt = \int_0^b 25e^t\,dt = \frac{25e^{0.5t}}{0.5}\Big|_0^b = 50e^{0.5b} - 50e^{0.5(0)}$$

$= 50e^{0.5b} - 50$. We are looking for when this result exceeds \$5,000, so

$50e^{0.5b} - 50 > 5000$ (solve for $e^{0.5b}$)

$50e^{0.5b} > 5050$

$e^{0.5b} > 101$ (take the natural log of both sides)

$\ln(e^{0.5b}) > \ln(101)$

$0.5b > 4.61512051684$

$b > 9.23024103368$

This means that during the tenth day, the sales will exceed \$5,000.

Ex. 5 A company determines that its marginal profit for a particular car is $p'(x) = -4x + 5600$ where x is the number of cars produced. Find the total profit from the production and sale of the 151st car to through the 500th car.

Solution:

We will need to integrate p'(x) from 150 to 500:

$$\int_{150}^{500} (-4x + 5600)\,dx = (-2x^2 + 5600x)\Big|_{150}^{500}$$

$= (-2(500)^2 + 5600(500)) - (-2(150)^2 + 5600(150))$

$= (2,300,000) - (795000) = 1,505,000$

The profit from the production and sale of the 151st car through the 500th car is \$1,505,000.

To find the average of a set of numbers, we add up the numbers and then divide by the number of numbers. We can think of the average over an interval as an accumulation divided by the length of the interval.

Average Value of a Function

Let f a continuous function over [a, b]. Then the average value of the function is:

$$\text{A.V.} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Let's look at an example:

Ex. 6 The number of bacteria present in a certain culture after t minutes of an experiment was $Q(t) = 4000e^{0.08t}$. What was the average number of bacteria present during the first five minutes of the experiment?

Solution:

Q(t) is a continuous function on [0, 5] so the average value is:

$$\frac{1}{5-0}\int_0^5 4000e^{0.08t}\,dt = \frac{1}{5}\cdot 4000\cdot\frac{e^{0.08t}}{0.08}\Big|_0^5 = 10000e^{0.08t}\Big|_0^5$$

$$= 10000(e^{0.08(5)} - e^0) = 10000(e^{0.4} - 1) \approx 4918.25$$

The average number of bacteria was about 4918.25 bacteria.

Ex, 7 From 5 am to 1 pm, the temperature outside was given by: $f(t) = -t^2 + 6t + 64$

where t is the time in hours after 5 am.

a) Find the average temperature.

b) At what time did a cold front start passing through the area? What was the temperature at that time?

c) Find the minimum temperature.

Solution:

a) Since 5 am corresponds to t = 0 and 1 pm corresponds to t = 8, then the domain of the function is [0, 8]. f(t) is a continuous function on [0, 8] so the average temperature is:

$$\frac{1}{8-0}\int_0^8 (-t^2 + 6t + 64)\,dt = \frac{1}{8}\bullet\left(-\frac{t^3}{3} + \frac{6t^2}{2} + 64t\right)\Big|_0^8$$

$$= \frac{1}{8}\left(-\frac{(8)^3}{3} + 3(8)^2 + 64(8)\right) - \frac{1}{8}\left(-\frac{(0)^3}{3} + 3(0)^2 + 64(0)\right)$$

$$= \frac{1}{8}\left(-\frac{512}{3} + 192 + 512\right) - 0 = \frac{1}{8}\left(\frac{1600}{3}\right) = \frac{200}{3} = 66\frac{2}{3}^\circ.$$

So, the average temperature is $66\frac{2}{3}^\circ$.

b) At the time the cold front started to pass through the area, the temperature was at its highest. So, we need to find the maximum temperature:

$f'(t) = \frac{d}{dt}[-t^2 + 6t + 64] = -2t + 6.$

Set $f'(t) = 0$ and solve:

$-2t + 6 = 0$

$-2t = -6$

$t = 3.$

So, $t = 3$ is are critical value.

Plugging $t = 3$ and the endpoints, $t = 0$ and $t = 8$ into the original function, we get:

$f(0) = -(0)^2 + 6(0) + 64 = 64$

$f(3) = -(3)^2 + 6(3) + 64 = -9 + 18 + 64 = 73$

$f(8) = -(8)^2 + 6(8) + 64 = -64 + 48 + 64 = 48$

The maximum temperature was 73°. It occurred at $t = 3$ which corresponds to 8 am.

c) Using our results from part b, the minimum temperature was 48°. It occurred at $t = 8$ which corresponds to 1 pm.