Section 5.4 - Properties of Definite Integrals

There are three properties are useful when working with definite integrals. Let us look at the first two properties:

1) $\int_a^b k\bullet f(x)\,dx = k\bullet\int_a^b f(x)\,dx$ where k is a constant

2) $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

The first property says if we are integrating a constant times a function, we can move the constant out in front of the integral and then integrate the function. The second property says we can integrate term by term. For the third property that we want to examine, if we are integrating over an interval, we can integrate the function over each piece of an interval:

3) If a < c < b, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

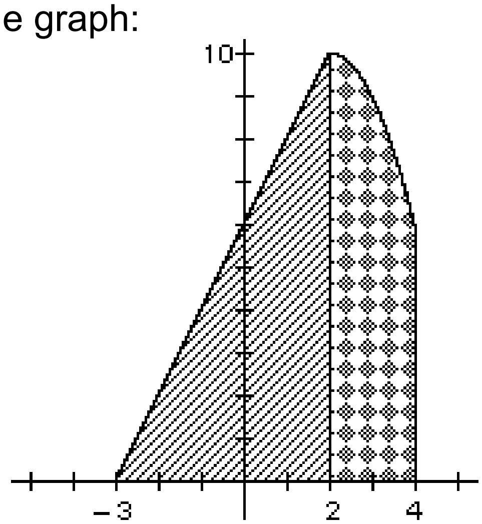
Since c is between a and b, we can first integrate from a to c and then from c to b and then add the two results. Where this is useful is when the function "changes" over an interval.

Ex. 1 Find the area under the graph of f(x) from – 3 to 4

where $f(x) = \begin{cases} 2x + 6, & \text{if } x < 2 \\ -x^2 + 4x + 6, & \text{if } x \geq 2 \end{cases}$

Solution:

Notice that the function "changes" at x = 2. Let's look at the graph:



To find the area under the curve, we will first integrate from – 3 to 2 and then integrate from 2 to 4. Afterwards, we we will add the two results to find the total area under f from – 3 to 4.

$$\int_{-3}^{4} f(x)\,dx = \int_{-3}^{2} f(x)\,dx + \int_{2}^{4} f(x)\,dx.$$

But $f(x) = 2x + 6$ from – 3 to 2 and $f(x) = -x^2 + 4x + 6$ from 2 to 4, so

$$\int_{-3}^{4} f(x)\,dx = \int_{-3}^{2} f(x)\,dx + \int_{2}^{4} f(x)\,dx$$

$$= \int_{-3}^{2} (2x + 6)\,dx + \int_{2}^{4} (-x^2 + 4x + 6)\,dx$$

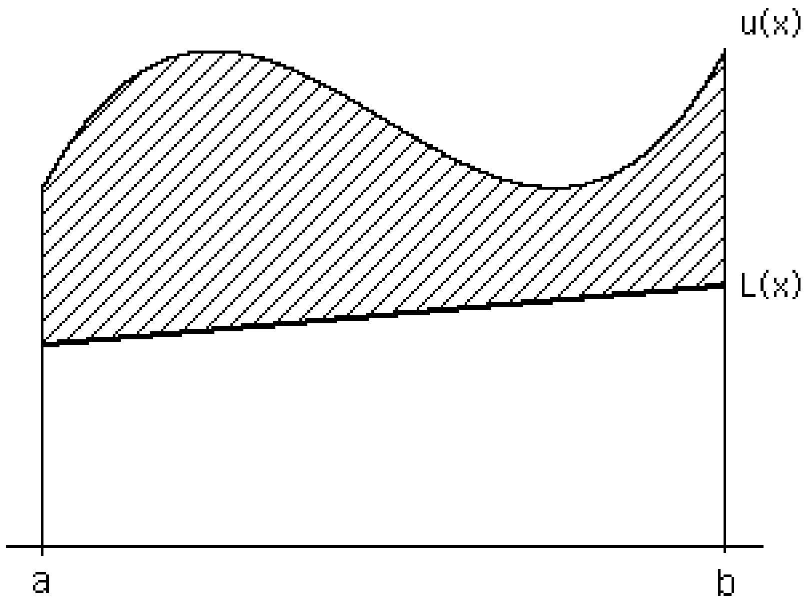
$$= [x^2 + 6x] \Big|_{-3}^{2} + \left(-\frac{x^3}{3} + 2x^2 + 6x\right) \Big|_{2}^{4}$$

$$= [(2)^2 + 6(2)] - [(-3)^2 + 6(-3)]$$
$$+ \left(-\frac{(4)^3}{3} + 2(4)^2 + 6(4)\right) - \left(-\frac{(2)^3}{3} + 2(2)^2 + 6(2)\right)$$

$$= [16] - [-9] + \left(-\frac{64}{3} + 56\right) - \left(-\frac{8}{3} + 20\right)$$

$$= 16 + 9 + 56 - 20 - \frac{64}{3} + \frac{8}{3} = \frac{127}{3} = 42\frac{1}{3} \text{ sq. units.}$$

We can apply our ideas of the definite integral to find the area between two curves on a specified interval. Let u(x) represent the upper curve and L(x) represent the lower curve on the interval [a, b]:



The area between the two curves is the area under curve u minus the area under curve L. But, the area of u(x) is

$\int_a^b u(x)\,dx$ and the area under L(x) is $\int_a^b L(x)\,dx$. Thus, the area between the two curves is $\int_a^b u(x)\,dx - \int_a^b L(x)\,dx$ or $\int_a^b [u(x) - L(x)]\,dx$ (Just using the fact we can integrate term by term backwards). Let's formalize this:

<u>Area Between Two Curves:</u>
If u(x) and L(x) are continuous functions and u(x) ≥ L(x) on [a, b], then the area between the two curves is:

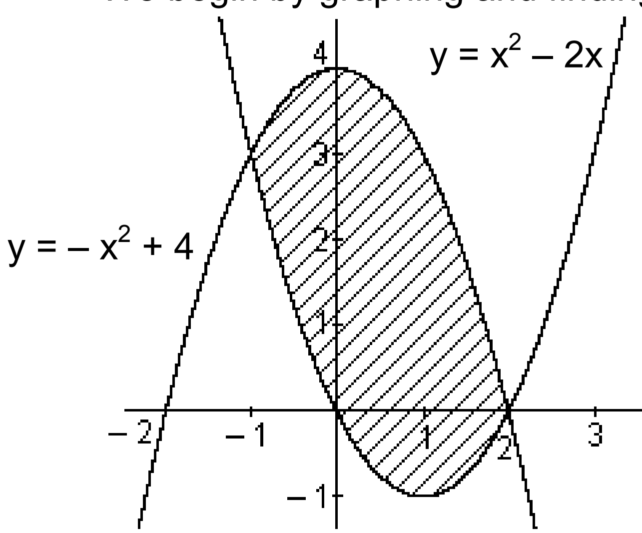
$$\int_a^b [u(x) - L(x)]\,dx$$

<u>Sketch the region R and then use calculus to find the area of R</u>:

Ex. 2 R is the region bounded by the curves $y = x^2 - 2x$ and $y = -x^2 + 4$.

<u>Solution:</u>

We begin by graphing and finding where they intersect:



Setting the two functions equal to each other and solve yields:

$x^2 - 2x = -x^2 + 4$
$2x^2 - 2x - 4 = 0$
$2(x^2 - x - 2) = 0$
$2(x - 2)(x + 1) = 0$
$x = 2$ and $x = -1$

$y\big|_{x=2} = 0$ and $y\big|_{x=-1} = 3$

The upper curve u(x) is $-x^2 + 4$ and the lower curve is $L(x) = x^2 - 2x$ on the interval [– 1, 2]. Thus, the area of R is:

$$\int_a^b [u(x) - L(x)]\,dx = \int_{-1}^{2} [-x^2 + 4 - (x^2 - 2x)]\,dx$$

$= \int_{-1}^{2} [-2x^2 + 2x + 4]\, dx = [-\frac{2}{3}x^3 + x^2 + 4x] \Big|_{-1}^{2}$

$= [-\frac{2}{3}(2)^3 + (2)^2 + 4(2)] - [-\frac{2}{3}(-1)^3 + (-1)^2 + 4(-1)]$

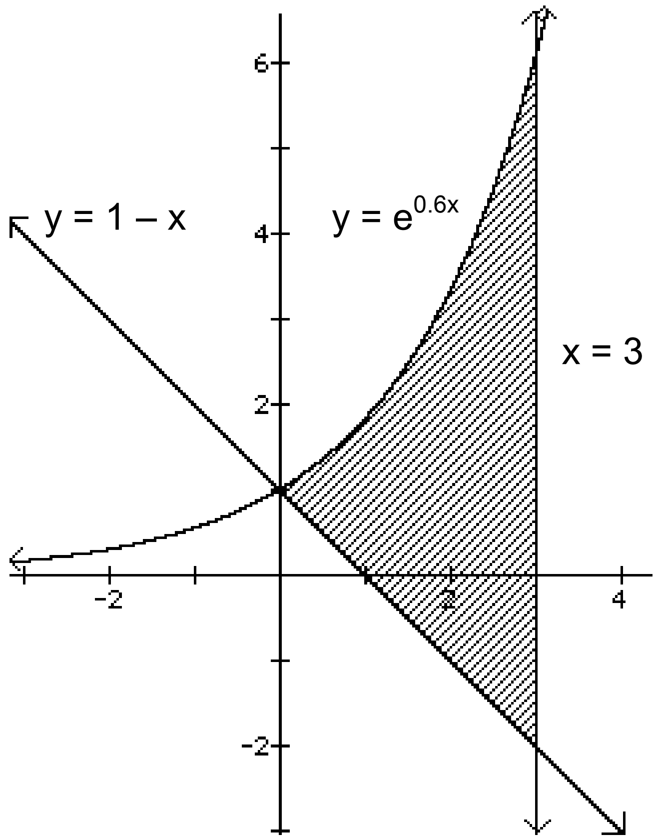
$= [-\frac{16}{3} + 4 + 8] - [\frac{2}{3} + 1 - 4] = -\frac{16}{3} + 12 - \frac{2}{3} + 3$

$= -6 + 15 = 9$ sq. units.

Ex. 3 R is the region bounded by the curves $x = 3$, $y = 1 - x$ and $y = e^{0.6x}$.

Solution:

We begin by graphing and finding where they intersect:



Notice that the curves $y = e^{0.6x}$ and $y = 1 - x$ intersect at the point (0, 1). Thus, the upper curve is $u(x) = e^{0.6x}$ and the lower curve is $L(x) = 1 - x$. The region starts at the point (0, 1) and ends at the line $x = 3$. Hence, we will be integrating over the interval [0, 3].

$\int_{a}^{b} [u(x) - L(x)]\, dx = \int_{0}^{3} [e^{0.6x} - (1 - x)]\, dx$

$= \int_{0}^{3} [e^{0.6x} - 1 + x]\, dx = [\frac{e^{0.6x}}{0.6} - x + \frac{x^2}{2}] \Big|_{0}^{3}$

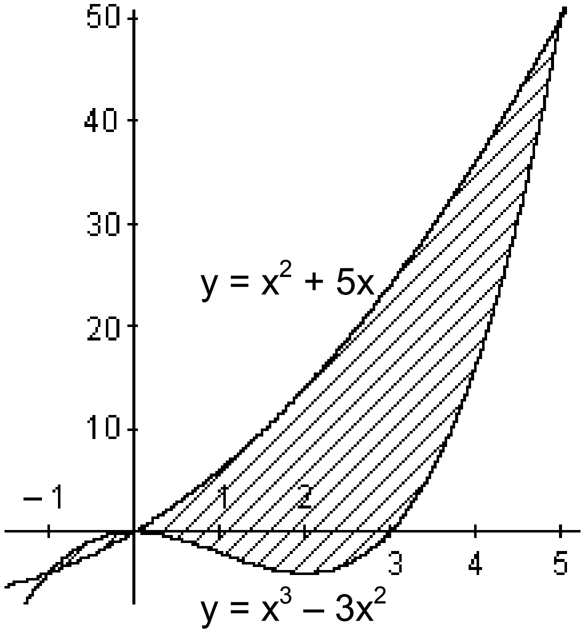
$= [\frac{e^{0.6(3)}}{0.6} - (3) + \frac{(3)^2}{2}] - [\frac{e^{0.6(0)}}{0.6} - (0) + \frac{(0)^2}{2}]$

$\approx [10.082745771 - 3 + 4.5] - [\frac{5}{3}] \approx 9.9160791$ sq. units.

Ex. 4 R is the region bounded by the curves $y = x^3 - 3x^2$ and $y = x^2 + 5x$.

<u>Solution:</u>

We begin by finding where they intersect & graph:



Setting the two functions equal to each other and solve yields:

$x^3 - 3x^2 = x^2 + 5x$

$x^3 - 4x^2 - 5x = 0$

$x(x^2 - 4x - 5) = 0$

$x(x - 5)(x + 1) = 0$

$x = 0,\ x = 5,\ \&\ x = -1$

$y\Big|_{x=5} = 50,\ \ y\Big|_{x=-1} = -4,$

and $y\Big|_{x=0} = 0.$

In [– 1, 0], $y = x^3 - 3x^2$ is the upper curve and in [0, 5], $y = x^2 + 5x$ is the upper curve. So, we will need to do this in two parts:

$$\int_{-1}^{0} [x^3 - 3x^2 - (x^2 + 5x)]dx + \int_{0}^{5} [x^2 + 5x - (x^3 - 3x^2)]dx$$

$$= \int_{-1}^{0} [x^3 - 4x^2 - 5x]dx + \int_{0}^{5} [-x^3 + 4x^2 + 5x]dx$$

$$= \left[\frac{x^4}{4} - \frac{4x^3}{3} - \frac{5x^2}{2}\right]\Bigg|_{-1}^{0} + \left[-\frac{x^4}{4} + \frac{4x^3}{3} + \frac{5x^2}{2}\right]\Bigg|_{0}^{5}$$

$$= [0] - \left[\frac{(-1)^4}{4} - \frac{4(-1)^3}{3} - \frac{5(-1)^2}{2}\right] + \left[-\frac{(5)^4}{4} + \frac{4(5)^3}{3} + \frac{5(5)^2}{2}\right] - [0]$$

$$= -\frac{1}{4} - \frac{4}{3} + \frac{5}{2} - \frac{625}{4} + \frac{500}{3} + \frac{125}{2} = -\frac{626}{4} + \frac{496}{3} + \frac{130}{2}$$

$$= 73\frac{5}{6} \text{ square units.}$$